

On the convergence of the energies
and the convergence of almost
minimizers in thin infinite magnetic
wires

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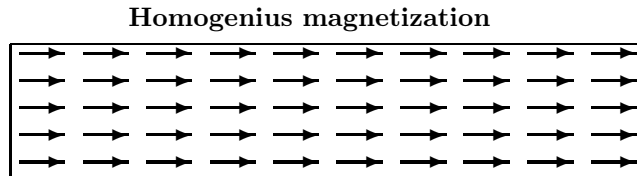
Abstract

We study static 180 degree domain walls in infinite magnetic wires with a bounded, centrally symmetric and simply-connected cross sections with a piecewise C^2 boundary. We prove an existence of global minimizers for the energy of micromagnetics and a Γ -convergence for the energies. We prove as well a rate of convergence for the minimal energies and a convergence of almost minimizers.

Keywords: Micromagnetics; Nanowires; Magnetization reversal; Domain wall

1 Introduction

It has been suggested in [1] that magnetic nanowires can be effectively used as storage devices. It is known that the magnetization pattern reversal time is closely related to the writing and reading speed of such a device, thus it is crucial to understand the magnetization reversal and switching processes. Several groups have numerically observed two different magnetization modes in magnetic nanowires. In [13] the magnetization reversal process has been studied numerically in cobalt nanowires by the Landau-Lishitz-Gilbert equation. Two different domain wall types were observed. For thin wires cobalt wires with 10nm in diameter the transverse mode has been observed: the magnetization is constant on each cross section and is moving along the wire. The gyromagnetic precession limits the domain wall velocity. The domain wall velocity is an increasing function of the Gilbert damping constant α . For thick wires, with diameters bigger than 20nm the vortex wall has been observed: the magnetization is approximately tangential to the boundary and forms a vortex which propagates along the wire. In this case the domain wall velocity is a decreasing function of the Gilbert damping constant α . In [15] the magnetization reversal process has been studied both numerically and experimentally. By considering a conical type wire so that the diameter of the cross section increases very slowly they observed the magnetization switching from the vortex wall to the transverse at a critical diameter, as the domain wall was moving along the wire. The results in [13] and [15] are very similar: in thin wires the transverse wall occurs, while in thick wires the vortex wall occurs. When a homogenous external field is applied in the axial direction of the wire facing the homogenous magnetization direction (see Fig. 1), then at a critical strength the reversal of the magnetization typically starts at one end of the wire creating a domain wall, which moves along the wire. The domain wall separates the reversed and the not yet reversed parts of the wire (see Fig. 1).



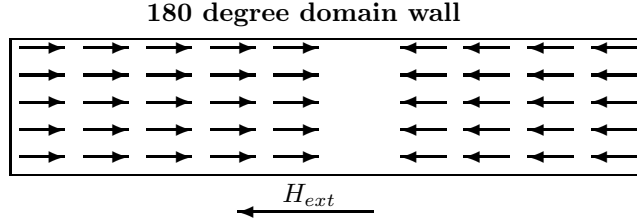


Figure 1.

In Figure 2 one can see the transverse and the vortex wall longitudinal and cross section pictures for wires with a rectangular cross section.

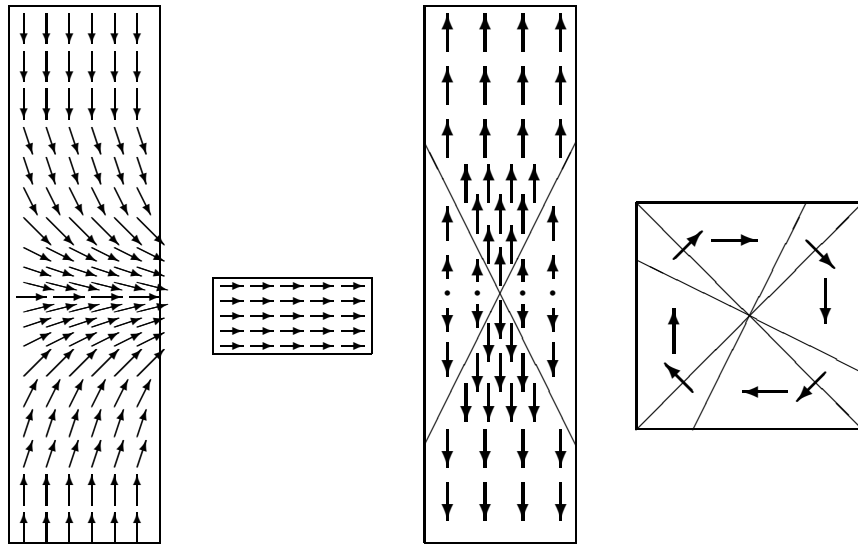


Figure 2.

The transverse wall

The vortex wall

It has been observed that there is a distinctive crossover between two different modes, which occurs at a critical diameter of the wire. It has been suggested that the magnetization switching process can be understood by analyzing the micromagnetics energy minimization problem for different diameters of the cross section. K. Kühn studied 180 degree static domain walls in magnetic wires with circular cross sections. She proved rigorously using variational methods in [19] that indeed the transverse mode must occur in thin magnetic wires. She also showed why the magnetization energy approximation by the energy

$$E(m) = \pi \|\partial_x m\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2} (\|m_y\|_{L^2(\mathbb{R})}^2 + \|m_z\|_{L^2(\mathbb{R})}^2)$$

done by Nakatani and Thiaville in [23] is legitimate. She proved a rate of convergence for the minimal energies, namely if E_R is the minimal energy in

the wire with a diameter $2R$ then

$$\left| \frac{E_R}{R^2} - 2\sqrt{2}\pi \right| \leq CR^2 |\ln R|.$$

It is also shown in [19] that for thick wires the vortex wall has the optimal energy scaling and that the minimal energy scales like $R^2\sqrt{\ln R}$. K. Kühn then studied the regularity of the minimizers, the dynamics of the transverse walls and the dynamics of the vortex walls in [20], [21] and [22] respectively. In this paper we study the 180 degree static domain walls in magnetic wires with arbitrary bounded, centrally symmetric and simply-connected cross sections with a piecewise C^2 boundary. We show that all the results obtained in [19] hold in this more general setting. Moreover, for a class of domains we prove the convergence of almost minimizers. We believe that our results obtained in this work can be used to study the dynamics of transverse walls in thin wires like it is done in [21].

In the theory of micromagnetics to any domain $\Omega \in \mathbb{R}^3$ and a unit vector field (called magnetization) $m: \Omega \rightarrow \mathbb{S}^2$ with $m = 0$ in $\mathbb{R}^3 \setminus \Omega$ the energy of micromagnetics is assigned:

$$E(m) = A_{ex} \int_{\Omega} |\nabla m|^2 + K_d \int_{\mathbb{R}^3} |\nabla u|^2 + Q \int_{\Omega} \varphi(m) - 2 \int_{\Omega} H_{ext} \cdot m,$$

where A_{ex} , K_d , Q are material parameters, H_{ext} is the externally applied magnetic field, φ is the anisotropy energy density and u is obtained from Maxwell's equations of magnetostatics, i.e., u is a weak solution of

$$\Delta u = \operatorname{div} m \quad \text{in} \quad \mathbb{R}^3.$$

According to micromagnetics, stable magnetization patterns are described by the minimizers of the micromagnetic energy functional.

2 The problem setting

Assume $\Omega = \mathbb{R} \times \omega$, where $\omega \subset \mathbb{R}^2$ is a bounded, centrally symmetric and simply-connected domain with a piecewise C^2 boundary. We consider the energy of micromagnetics without an external field and anisotropy energy:

$$E(m) = A_{ex} \int_{\Omega} |\nabla m|^2 + K_d \int_{\mathbb{R}} |\nabla u|^2.$$

By scaling of all coordinates one can achieve the situation where $A_{ex} = K_d$, so we will henceforth assume that $A_{ex} = K_d = 1$. Denote

$$A(\Omega) = \{m: \Omega \rightarrow \mathbb{S}^2 : m \in H_{loc}^1(\Omega), E(m) < \infty\}.$$

We are interested in 180 degree domain walls, so set

$$\tilde{A}(\Omega) = \{m: \Omega \rightarrow \mathbb{S}^2 : m - \bar{e} \in H^1(\Omega)\},$$

where

$$\bar{e}(x, y, z) = \begin{cases} (-1, 0, 0) & \text{if } x < -1 \\ (x, 0, 0) & \text{if } -1 \leq x \leq 1 \\ (1, 0, 0) & \text{if } 1 < x \end{cases}$$

Roughly speaking we are considering the set of all magnetizations that satisfy $\lim_{x \rightarrow \pm\infty} m(x, y, z) = \pm \vec{e}_x$ for all y and z . The objective of this work will be studying the minimization problem

$$\inf_{m \in \tilde{A}(\Omega)} E(m) \quad (1)$$

3 Notation

The letter $\xi = (\xi_1, \xi_2, \xi_3)$ denotes a point in \mathbb{R}^3 , a map $f: \Omega \rightarrow \mathbb{R}^3$ will have the components f_x, f_y, f_z , i.e., $f = (f_x, f_y, f_z)$. Set

$$\begin{aligned} \text{diam}(\omega) &= d, \\ E_{ex}(m) &= \int_{\Omega} |\nabla m|^2, \quad E_{mag}(m) = \int_{\mathbb{R}^3} |\nabla u|^2, \\ A_x(\Omega) &= \{m \in A(\Omega) : m = m(x)\}, \\ M(\Omega) &= \{m: \Omega \rightarrow \mathbb{R}^3 : m \in H_{loc}^1(\Omega), |m| \leq M, m(\xi) = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Omega}\}, \\ M_x(\Omega) &= \{m \in M(\Omega) : m = m(x)\}. \end{aligned}$$

For any $m \in M(\Omega)$ the divergence of m consists of the body charges v and the surface charges s , i.e.,

$$\begin{aligned} v(\xi) &= \begin{cases} -\text{div} m & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \Omega \end{cases} \\ s(\xi) &= \begin{cases} m(\xi) \cdot \nu(\xi) & \text{on } \partial\Omega \\ 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega \end{cases} \end{aligned}$$

where $\nu(\xi)$ is the outward unit normal to the boundary of Ω at point ξ . Recall that the map u is the weak solution of

$$\Delta u = \text{div} m \quad \text{in } \mathbb{R}^3 \quad (2)$$

if and only if

$$\nabla u \in L^2(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^3} m \cdot \nabla \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3) \quad (3)$$

which is itself equivalent to

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_{\Omega} v \cdot \varphi + \int_{\partial\Omega} s \cdot \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \quad (4)$$

We decompose $u = u_v + u_s$ so that

$$\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla \varphi = \int_{\Omega} v \cdot \varphi, \quad \int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi = \int_{\partial\Omega} s \cdot \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3).$$

Set

$$E_v = \int_{\mathbb{R}^3} |\nabla u_v|^2, \quad E_s = \int_{\mathbb{R}^3} |\nabla u_s|^2, \quad E_{vs} = \int_{\mathbb{R}^3} \nabla u_v \cdot \nabla u_s.$$

Next we define

$$\bar{m}(x, y, z) = \frac{1}{|\omega|} \int_{\omega} m \, dy \, dz, \quad (x, y, z) \in \Omega.$$

Like m we extend \bar{m} as 0 outside $\bar{\Omega}$. It is evident that if m is weakly differentiable in x then so is \bar{m} and

$$\partial_x \bar{m}(x, y, z) = \frac{1}{|\omega|} \int_{\omega} \partial_x m(x, y_1, z_1) \, dy_1 \, dz_1, \quad (x, y, z) \in \Omega.$$

4 Preliminaries

In this section we give several descriptions of the set $A(\Omega)$. We start with a lemma that states the finiteness of the norm $\|s\|_{L^2(\partial\omega)}$

Lemma 4.1. *For any $m \in M_x(\Omega)$ with $E(m) < \infty$ one has*

$$\|s\|_{L^2(\partial\omega)} < \infty.$$

Proof. The idea of the proof is choosing suitable test functions φ in (4). Assume that the equation of $\partial\omega$ is $r = \rho(\theta)$ in polar coordinates. Fix any $R > 0$ and consider a cutoff function ψ that satisfies the following conditions:

$$\begin{aligned} 0 \leq \psi(x) \leq 1, & \quad x \in [-R-1, R+1], & \quad \text{supp}(\psi) \subset [-R-1, R+1], \\ \psi(x) = 1, & \quad x \in [-R, R], \\ |\psi'(x)| \leq 2, & \quad x \in [-R-1, R+1]. \end{aligned}$$

Consider now the test function

$$\varphi(x, y, z) = \psi(x) \cdot \min(2r - \rho(\theta), 3\rho(\theta) - 2r) \cdot s(\rho(\theta) \cos \theta, \rho(\theta) \sin \theta),$$

where

$$x \in [-R-1, R+1], \quad y = r \cos \theta, \quad z = r \sin \theta, \quad r \in \left[\frac{\rho(\theta)}{2}, \frac{3\rho(\theta)}{2} \right].$$

In the proof constants C can depend only on ω . We place ω such that it be symmetric with respect to the origin. One can estimate the summands in (4) by direct calculation and the Schwartz inequality as follows

$$\begin{aligned} |\nabla \varphi|^2 &\leq C(|s|^2 + |\nabla s|^2) \leq C(|s|^2 + |\nabla m|^2), \\ \int_{\partial\Omega} \varphi \cdot s &\geq \int_{[-R, R] \times \partial\omega} \rho(\theta) |s|^2 \geq \min_{\theta \in [0, 2\pi]} \rho(\theta) \int_{[-R, R] \times \partial\omega} |s|^2, \\ \left| \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right| &\leq C \|\nabla u\|_{L^2(\mathbb{R})} \sqrt{\|s\|_{L^2([-R, R] \times \omega)}^2 + \|\nabla m\|_{L^2(\Omega)}^2 + 1}, \\ \left| \int_{\Omega} v \cdot \varphi \right| &\leq C \|\nabla m\|_{L^2(\Omega)} (\|s\|_{L^2([-R, R] \times \omega)} + 1). \end{aligned}$$

Since ω is open and simply-connected, the origin does not belong to its boundary, thus $\min_{\theta \in [0, 2\pi]} \rho(\theta) > 0$ and we discover from (4)

$$\|s\|_{L^2([-R,R] \times \omega)}^2 \leq C(\|s\|_{L^2([-R,R] \times \omega)} + 1),$$

from where we arrive at

$$\|s\|_{L^2([-R,R] \times \omega)} \leq C.$$

Since R is arbitrary we get $\|s\|_{L^2(\partial\Omega)} \leq C$ as claimed. \square

Corollary 4.2. *Any $m \in M_x(\Omega)$ with a finite energy satisfies*

$$\|m_y\|_{L^2(\mathbb{R})}^2 + \|m_z\|_{L^2(\mathbb{R})}^2 < \infty.$$

Proof. It follows from Lemma 4.1 and the fact that $\partial\omega$ is piecewise C^2 . \square

Lemma 4.3. *For any vector fields $m_1, m_2 \in M(\Omega)$ with finite energies*

- (i) $E_{mag}(m_1 + m_2) \leq 2(E_{mag}(m_1) + E_{mag}(m_2))$
- (ii) $|E_{mag}(m_1) - E_{mag}(m_2)| \leq E_{mag}(m_1 - m_2) + 2\sqrt{E_{mag}(m_1)E_{mag}(m_1 - m_2)}$
- (iii) $|E_{mag}(m_1) - E_{mag}(m_2)| \leq \|m_1 - m_2\|_{L^2(\Omega)}^2 + 2\|m_1 - m_2\|_{L^2(\Omega)}\sqrt{E_{mag}(m_1)}$

Proof. Let u_1 and u_2 satisfy $\Delta u_1 = \operatorname{div} m_1$ and $\Delta u_2 = \operatorname{div} m_2$. Then $\Delta(u_1 + u_2) = \operatorname{div}(m_1 + m_2)$, thus

$$E_{mag}(m_1 + m_2) = \|\nabla(u_1 + u_2)\|_{L^2(\Omega)}^2 \leq 2(\|\nabla u_1\|_{L^2(\Omega)}^2 + \|\nabla u_2\|_{L^2(\Omega)}^2).$$

We have

$$\begin{aligned} |E_{mag}(m_1) - E_{mag}(m_2)| &= \left| \|\nabla u_1\|_{L^2(\Omega)}^2 - \|\nabla u_2\|_{L^2(\Omega)}^2 \right| \\ &\leq \|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 + 2\|\nabla u_1\|_{L^2(\Omega)}\|\nabla(u_1 - u_2)\|_{L^2(\Omega)}. \end{aligned}$$

The third statement is a consequence of the second one and the inequality $\|\nabla u\|_{L^2(\Omega)}^2 \leq \|m\|_{L^2(\Omega)}^2$. \square

Lemma 4.4. *For any $m \in A(\Omega)$ one has*

(i)

$$\int_{\omega} (|m|^2 - |\bar{m}|^2) = \int_{\omega} |m - \bar{m}|^2 \leq C_p d^2 \int_{\omega} |\nabla_{yz} m| \quad \text{for all } x \in \mathbb{R},$$

where C_p is the Poincaré constant for ω .

(ii) $E_{ex}(\bar{m}) + E_{ex}(m - \bar{m}) = E_{ex}(m)$

Proof. We have for any $x \in \mathbb{R}$

$$\int_{\omega} (m - \bar{m}) = \int_{\omega_x} m - |\omega| \cdot \bar{m}(x) = 0,$$

thus

$$\begin{aligned}\int_{\omega} |m|^2 &= \int_{\omega} |\bar{m}|^2 + \int_{\omega} |m - \bar{m}|^2 + 2\bar{m}(x) \int_{\omega} (m - \bar{m}) \\ &= \int_{\omega} |\bar{m}|^2 + \int_{\omega} |m - \bar{m}|^2.\end{aligned}$$

Taking into account the fact that the weak derivative of the average function is the average of the original function's weak derivative we get the second identity. \square

Corollary 4.5. *For any $m \in A(\Omega)$ and $x \in \mathbb{R}$*

$$\int_{\omega} |\bar{m}|^2 \leq \int_{\omega} |m|^2 \leq \int_{\omega} |\bar{m}|^2 + C_p d^2 \int_{\omega} |\nabla_{yz} m|^2.$$

Corollary 4.6. *If $m \in A(\Omega)$ then*

- (i) $|E_{mag}(m) - E_{mag}(\bar{m})| \leq d(C_p d + 2\sqrt{C_p})E(m),$
- (ii) $E(\bar{m}) \leq E(m)(1 + d(C_p d + 2\sqrt{C_p}))E(m).$

Proof. It is a consequence of Lemmas 4.3, 4.4 and the inequality

$$E_{ex}(\bar{m}) = \|\nabla \bar{m}\|_{\Omega}^2 = \int_{\Omega} |\partial_x \bar{m}|^2 \leq \int_{\Omega} |\partial_x m|^2 \leq E_{ex}(m).$$

\square

Lemma 4.7. *Let $m \in A(\Omega)$ and let $\alpha, \beta \in \mathbb{R}$ such that $-1 < \alpha < \beta < 1$. Assume \mathfrak{R} is family of disjoint intervals (a, b) satisfying the conditions*

$$\{\bar{m}_x(a), \bar{m}_x(b)\} = \{\alpha, \beta\} \quad \text{and} \quad |\bar{m}_x(x)| \leq \max(|\alpha|, |\beta|), \quad x \in (a, b).$$

Then,

$$\text{card}(\mathfrak{R}) \leq M(\alpha, \beta) \quad \text{and} \quad \sum_{(a,b) \in \mathfrak{R}} (b - a) \leq M(\alpha, \beta),$$

where $M(\alpha, \beta)$ is a constant depending on α, β, ω and $E(m)$. Moreover, $\lim_{x \rightarrow \pm\infty} |\bar{m}_x| = 1$.

Proof. We first prove that the sum of the lengths of the intervals in \mathfrak{R} is bounded. Denote $\max(|\alpha|, |\beta|) = \rho$. The function \bar{m} is a one variable weakly differentiable function therefore it is locally absolutely continuous in \mathbb{R} . For any $(a, b) \in \mathfrak{R}$, we have by Corollary 4.5,

$$\begin{aligned}|\omega|(b - a) &= \int_{(a,b) \times \omega} |m|^2 \\ &\leq \int_{(a,b) \times \omega} |\bar{m}|^2 + C_p d^2 \int_{(a,b) \times \omega} |\nabla_{yz} m|^2 \\ &\leq \rho^2 |\omega|(b - a) + \int_{(a,b) \times \omega} (\bar{m}_y^2 + \bar{m}_z^2) + C_p d^2 \int_{(a,b) \times \omega} |\nabla m|^2.\end{aligned}$$

Summing up the obtained inequality for all $(a, b) \in \mathfrak{R}$ and denoting

$$\Sigma = \bigcup_{(a,b) \in \mathfrak{R}} (a, b) \times \omega$$

we get

$$\begin{aligned} |\omega| \cdot \sum_{(a,b) \in \mathfrak{R}} (b-a) &\leq \rho^2 |\omega| \cdot \sum_{(a,b) \in \mathfrak{R}} (b-a) + \int_{\Sigma} (\bar{m}_y^2 + \bar{m}_z^2) + C_p d^2 \int_{\Sigma} |\nabla m|^2 \\ &\leq \rho^2 |\omega| \sum_{(a,b) \in \mathfrak{R}} (b-a) + \int_{\Omega} (\bar{m}_y^2 + \bar{m}_z^2) + C_p d^2 \int_{\Omega} |\nabla m|^2. \end{aligned}$$

By virtue of Corollary 4.6 the average \bar{m} has a finite energy, thus Corollary 4.1 gives

$$\int_{\Omega} (\bar{m}_y^2 + \bar{m}_z^2) \leq C_1,$$

for some C_1 . Therefore we obtain

$$|\omega| \cdot \sum_{(a,b) \in \mathfrak{R}} (b-a) \leq \rho^2 |\omega| \sum_{(a,b) \in \mathfrak{R}} (b-a) + C_1 + C_p d^2 E(m).$$

Finally we get

$$\sum_{(a,b) \in \mathfrak{R}} (b-a) \leq \frac{C_1 + C_p d^2 E(m)}{|\omega|(1-\rho^2)}. \quad (5)$$

Now we prove an upper bound on the number of the entries of \mathfrak{R} . For any point $(y, z) \in \omega$ and any interval $(a, b) \in \mathfrak{R}$ we have

$$\int_a^b |\partial_x m_x(x, y, z)|^2 dx \geq \frac{1}{b-a} \left(\int_a^b |\partial_x m_x(x, y, z)| dx \right)^2.$$

Integrating over ω we get

$$\begin{aligned} \int_{(a,b) \times \omega} |\partial_x m_x|^2 d\xi &\geq \frac{1}{b-a} \int_{\omega} \left(\int_a^b |\partial_x m_x(x, y, z)| dx \right)^2 dy dz \\ &\geq \frac{1}{b-a} \int_{\omega} |m_x(a, y, z) - m_x(b, y, z)|^2 dy dz \\ &\geq \frac{1}{|\omega|(b-a)} \left(\int_{\omega} |m_x(a, y, z) - m_x(b, y, z)| dy dz \right)^2 \\ &\geq \frac{1}{|\omega|(b-a)} \left(\int_{\omega} (m_x(a, y, z) - m_x(b, y, z)) dy dz \right)^2 \\ &= \frac{1}{|\omega|(b-a)} \left(|\omega|(\bar{m}_x(a) - \bar{m}_x(b)) \right)^2 \\ &= \frac{|\omega|(\alpha - \beta)^2}{b-a}, \end{aligned}$$

thus

$$\int_{(a,b) \times \omega} |\partial_x m_x|^2 d\xi \geq \frac{|\omega|(\alpha - \beta)^2}{b-a}.$$

Summing up the last inequality for all $(a, b) \in \mathfrak{R}$ we arrive at

$$\begin{aligned} \sum_{(a,b) \in \mathfrak{R}} \frac{1}{b-a} &\leq \frac{1}{|\omega|(\alpha-\beta)^2} \int_{\Sigma} |\partial_x m_x|^2 d\xi \\ &\leq \frac{1}{|\omega|(\alpha-\beta)^2} \int_{\Omega} |\nabla m|^2 d\xi \\ &\leq \frac{E(m)}{|\omega|(\alpha-\beta)^2}, \end{aligned}$$

thus

$$\sum_{(a,b) \in \mathfrak{R}} \frac{1}{b-a} \leq \frac{E(m)}{|\omega|(\alpha-\beta)^2}. \quad (6)$$

Adding (5) and (6) we obtain

$$\sum_{(a,b) \in \mathfrak{R}} \left(\frac{1}{b-a} + b-a \right) \leq \frac{1}{|\omega|} \left(\frac{E(m)}{(\alpha-\beta)^2} + \frac{C_1 + C_p d^2 E(m)}{1-\rho^2} \right) := M(\alpha, \beta).$$

Coupling now the last inequality and the fact that for any $(a, b) \in \mathfrak{R}$ the inequality $\frac{1}{b-a} + b-a \geq 2$ holds we obtain $M(\alpha, \beta) \geq 2N$ where N is the number of the entries of \mathfrak{R} and $M(\alpha, \beta)$ depends only on α , β , ω and $E(m)$. The first part is proved.

It is clear that

$$|\bar{m}_x(x)| = \frac{1}{|\omega|} \left| \int_{\omega} m_x(x, y, z) dy dz \right| \leq \frac{1}{|\omega|} \int_{\omega} |m_x(x, y, z)| dy dz \leq 1$$

thus

$$0 \leq 1 - \bar{m}_x^2(x) \leq 1, \quad x \in \mathbb{R}.$$

By virtue of Lemma 4.4 we have

$$\int_{\Omega} (1 - \bar{m}_x^2) d\xi \leq \int_{\Omega} (\bar{m}_y^2 + \bar{m}_z^2) d\xi + C_p d^2 E(m) < \infty, \quad (7)$$

thus

$$\int_{\mathbb{R}} (1 - \bar{m}_x^2) dx < \infty.$$

The integrand is continuous and positive thus for any $0 < \delta < 1$ and $N > 0$ there exists $x_{\delta} > N$ such that $|\bar{m}_x(x_{\delta})| > 1 - \frac{\delta}{2}$. Therefore there exists an increasing sequence $\{x_n\}$ such that $x_n \rightarrow \infty$ and $|\bar{m}_x(x_n)| > 1 - \frac{\delta}{2}$. Therefore for infinitely many indices n one has one of the following: $\bar{m}_x(x_n) > 1 - \frac{\delta}{2}$ or $\bar{m}_x(x_n) < -1 + \frac{\delta}{2}$. Assume that for a subsequence (not relabeled) we have $\bar{m}_x(x_n) > 1 - \frac{\delta}{2}$. We will prove that $\bar{m}_x(x) > 1 - \delta$ for all $x > N_{\delta}$ for some N_{δ} . Assume in the contrary that for an increasing sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ with $\tilde{x}_n \rightarrow \infty$ one has $\bar{m}_x(\tilde{x}_n) \leq 1 - \delta$. We construct an infinite family of disjoint intervals (a_n, b_n) such that the value of \bar{m}_x at one of the ends of (a_n, b_n) is less or equal than $1 - \delta$ and at the other end is bigger than $1 - \frac{\delta}{2}$ for all $n \in \mathbb{N}$. In the first step we take the smallest n such that $\tilde{x}_n > x_1$ and denote it by \tilde{n}_1 and take

$a_1 = x_1$, $b_1 = \tilde{x}_{\tilde{n}_1}$. In the second step we take the smallest n such that $x_n > b_1$ and denote it by n_2 and then we take the smallest n such that $\tilde{x}_n > x_{n_2}$ and denote it by \tilde{n}_2 and take $a_2 = x_{n_2}$ and $b_2 = \tilde{x}_{\tilde{n}_2}$. This process is continuable without a stop, thus the intervals (a_n, b_n) are constructed such that $\bar{m}_x(a_n) > 1 - \frac{\delta}{2}$ and $\bar{m}_x(b_n) < 1 - \delta$. Since \bar{m}_x is continuous in \mathbb{R} the new sequence of disjoint intervals $(\acute{a}_n, \acute{b}_n)$ where $\acute{a}_n = \sup\{x \in (a_n, b_n) \mid \bar{m}_x(x) \geq 1 - \frac{\delta}{2}\}$ and $\acute{b}_n = \inf\{x \in (\acute{a}_n, b_n) \mid \bar{m}_x(x) \leq 1 - \delta\}$ has the properties $\bar{m}_x(\acute{a}_n) = 1 - \frac{\delta}{2}$, $\bar{m}_x(\acute{b}_n) = 1 - \delta$ and $|\bar{m}_x(x)| \leq 1 - \frac{\delta}{2}$ for all $x \in [\acute{a}_n, \acute{b}_n]$. But this contradicts the first statement of the foregoing lemma. The same can be done for $-\infty$. \square

Remark 4.8. In the proof of Lemma 4.7 we have actually shown that for an arbitrary magnetization m the finiteness of the three norms

$$\|\nabla m\|_{L^2(\Omega)}, \quad \|\bar{m}_y\|_{L^2(\mathbb{R})}, \quad \|\bar{m}_z\|_{L^2(\mathbb{R})}$$

yields that \bar{m}_x has a constant sign at both $\pm\infty$.

The next theorem in the analog (for C^2 cross sections) of the necessity part of Theorem 16 in [19]. It describes the set of all magnetizations which are constant on each cross section and have a finite energy.

Lemma 4.9. If $m \in A$ then one of the four vector fields $m \pm \vec{e}_x$, $m \pm \vec{e}$ belongs to $H^1(\Omega)$.

Proof. For any $m \in A$ we have

$$E(m) = \int_{\Omega} |\nabla m|^2 d\xi + E_{mag} < \infty$$

thus $\nabla m \in L^2(\Omega)$. Note that $\nabla \vec{e}_x, \nabla \vec{e} \in L^2(\Omega)$, thus by the triangle inequality $\nabla(m \pm \vec{e}_x), \nabla(m \pm \vec{e}) \in L^2(\Omega)$. It remains to prove that one of the four functions $m \pm \vec{e}_x, m \pm \vec{e}$ belongs to $L^2(\Omega)$. Denote

$$\Omega_- = (-\infty, 0] \times \omega \quad \text{and} \quad \Omega_+ = [0, +\infty) \times \omega.$$

We have

$$\begin{aligned} \int_{\Omega_-} |m - \vec{e}_x|^2 d\xi &= \int_{\Omega_-} ((m_x - 1)^2 + m_y^2 + m_z^2) d\xi \\ &= 2 \int_{\Omega_-} (1 - m_x) d\xi \\ &= 2|\omega| \int_{-\infty}^0 (1 - \bar{m}_x) dx \end{aligned}$$

and similarly

$$\int_{\Omega_+} |m + \vec{e}_x|^2 d\xi = 2|\omega| \int_0^{\infty} (1 + \bar{m}_x) dx$$

It is now clear that $m \pm \vec{e}_x \in L^2(\Omega_-)$ if and only if $1 \pm \bar{m}_x \in L^1(-\infty, 0)$. Similarly we have that $m \pm \vec{e}_x \in L^2(\Omega_+)$ if and only if $1 \pm \bar{m}_x \in L^1(0, +\infty)$. According to Remark 4.8 the component \bar{m}_x has a constant sign at $\pm\infty$. Suppose

that $\bar{m}_x(x) \geq 0$ for $x \geq N \geq 0$. According to (7) we have that

$$\int_0^{+\infty} (1 - \bar{m}_x^2) dx < \infty$$

thus

$$\begin{aligned} \int_0^{+\infty} (1 - \bar{m}_x^2) dx &\geq \int_N^{+\infty} (1 - \bar{m}_x^2) dx \\ &= \int_N^{+\infty} (1 - \bar{m}_x)(1 + \bar{m}_x) dx \\ &\geq \int_N^{+\infty} (1 - \bar{m}_x) dx, \end{aligned}$$

and thus

$$\int_0^{+\infty} (1 - \bar{m}_x) dx \leq 2N + \int_N^{+\infty} (1 - \bar{m}_x) dx < \infty.$$

Similarly we can prove that if we had $\bar{m}_x(x) < 0$ for $x \geq N > 0$ for some N then $1 + \bar{m}_x \in L^1(0, +\infty)$. Obviously the same can be done for Ω_- , thus the theorem is proved. \square

5 The representation of E_s in Fourier space and the characterization theorem

In this section we will find the representation of E_s in Fourier space and we will also show that the inverse of Lemma 4.9 holds. Like it is done in [19] one can prove that if $m - \bar{e} \in H^1(\Omega)$ then the equation $\Delta u = \operatorname{div} m$ has a solution u with a finite L^2 -norm. It is very well known that if $m \in L^2(\Omega)$ then $\Delta u = \operatorname{div} m$ has a weak solution u with $\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq \|m\|_{L^2(\Omega)}$, which is basically the L^2 projection of m onto the closure of the subspace

$$\{\nabla \varphi : \varphi \in C_0^\infty(\mathbb{R}^3)\}$$

of gradients in the Hilbert space $L^2(\mathbb{R}^3)$.

Following Kühn like in [19] we consider for all $c^-, c^+ \in \mathbb{R}$ the function $\chi_{c^-}^{c^+} : \mathbb{R} \rightarrow \mathbb{R}^3$ such that

$$\chi_{c^-}^{c^+} = (e^{\operatorname{sign}(x)} \min(1, |x|), 0, 0)$$

and define the set

$$C(\Omega) = \{m : \Omega \rightarrow \mathbb{R}^3 : \exists c^-, c^+ \in \mathbb{R} \text{ such that } m - \chi_{c^-}^{c^+} \in H^1(\Omega)\}.$$

Recall that the fundamental solution of the Laplace equation in \mathbb{R}^3 is $\Gamma(\xi) = \frac{1}{4\pi|\xi|}$. Next we formulate the analog (for C^2 cross sections) of Lemma 4 in [19].

Theorem 5.1. For $m \in C(\Omega)$ define the maps $u_v, u_s, u: \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$\begin{aligned} u_v(\xi) &= \int_{\Omega} \Gamma(\xi - \xi_1) v(\xi_1) d\xi_1, \\ u_s(\xi) &= \int_{\partial\Omega} \Gamma(\xi - \xi_1) s(\xi_1) d\xi_1, \\ u(\xi) &= u_v(\xi) + u_s(\xi). \end{aligned}$$

Then the following statements hold:

(i) The maps u_v and u_s satisfy the equalities

$$\begin{aligned} \nabla u_v(\xi) &= \sum_{i \in \{x, y, z\}} \int_{\Omega} \partial_i \Gamma(\xi - \xi_1) v(\xi_1) \vec{e}_i d\xi_1 \quad \text{for all } \xi \in \mathbb{R}^3, \\ \nabla u_s(\xi) &= \sum_{i \in \{x, y, z\}} \int_{\partial\Omega} \partial_i \Gamma(\xi - \xi_1) s(\xi_1) \vec{e}_i d\xi_1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \partial\Omega, \\ \int_{\mathbb{R}^3} \nabla u_v \cdot \nabla \varphi &= \int_{\Omega} v \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi &= \int_{\partial\Omega} s \varphi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \end{aligned}$$

(ii) u is a weak solution of $\Delta u = \text{div} m$.

(iii) $\nabla u \in L^2(\mathbb{R}^3)$.

For a proof we refer to [19] mentioning that it depends on the regularity of $\partial\omega$ and not on the shape. Note that $\tilde{A}(\Omega) \subset C(\Omega)$ thus for any $m \in \tilde{A}(\Omega)$ we will hereafter consider the weak solution of $\Delta u = \text{div} m$ which is defined in Theorem 5.1. As a corollary we get a necessary and sufficient condition for a magnetization to have a finite energy which is the analog of Theorem 16 in [19].

Theorem 5.2. A magnetization $m: \Omega \rightarrow \mathbb{S}^2$ has a finite energy if and only if one of the four vector fields $m \pm \vec{e}_x, m \pm \vec{e}$ belongs to $H^1(\Omega)$.

Proof. The necessity is Theorem 4.9. To prove the sufficiency we note that if one of the four functions $m \pm \vec{e}_x, m \pm \vec{e}$ belongs to $H^1(\Omega)$ then $m \in C(\Omega)$ with $|c_i| = 1$, thus according to Theorem 5.1 we get $m \in A(\Omega)$. \square

Corollary 5.3. A magnetization m has a finite energy if and only if

$$\nabla m, m_y, m_z \in L^2(\Omega).$$

Proof. The necessity is a consequence of Theorem 5.2 and the inequality

$$\|m_y\|_{L^2(\Omega)}^2 + \|m_z\|_{L^2(\Omega)}^2 + \|\nabla m\|_{L^2(\Omega)}^2 \leq \min(\|m \pm \vec{e}_x\|_{H^1(\Omega)}^2, 4|\omega| + \|m \pm \vec{e}\|_{H^1(\Omega)}^2).$$

Assume now that $\nabla m, m_y, m_z \in L^2(\Omega)$. By Lemma 4.4 we have

$$\|\bar{m}_y\|_{L^2(\Omega)}^2 + \|\bar{m}_z\|_{L^2(\Omega)}^2 \leq \|m_y\|_{L^2(\Omega)}^2 + \|m_z\|_{L^2(\Omega)}^2 + C_p d^2 \|\nabla m\|_{L^2(\Omega)}^2 < \infty,$$

thus from (7) we get $\|1 - \bar{m}_x\|_{L^2(\Omega)} < \infty$. The rest can be done like in the proof of Theorem 4.9. \square

Corollary 5.4. For any magnetization $m \in \tilde{A}_x(\Omega)$ denote

$$m^*(x) = \begin{cases} m_x(x) + 1 & \text{if } x \in (-\infty, 0] \\ m_x(x) - 1 & \text{if } x \in (0, +\infty), \end{cases}$$

then $m^* \in L^2(\mathbb{R})$.

Proof. According to Remark 4.8 we have that there exists $N > 0$ such that $m_x(x) > 0$ in $[N, +\infty)$. We have by virtue of Corollary 5.3,

$$\begin{aligned} \int_0^{+\infty} (m^*(x))^2 dx &\leq 4N + \int_N^{+\infty} (1 - m_x^2(x)) dx \\ &= 4N + \int_N^{+\infty} (m_y^2(x) + m_z^2(x)) dx \\ &< \infty. \end{aligned}$$

□

We consider now the functional E_{mag} for the magnetisations which are constant on each cross section, i.e., for $m \in A_x$.

Lemma 5.5. For any $m \in A_x$ the gradients ∇u_v and ∇u_s are orthogonal in $L^2(\mathbb{R}^3)$.

Proof. Since v is independent of y and $s(x, y, z) = -s(x, -y, -z)$ then we get by direct calculation that $E_{vs} = 0$.

□

Thus for $m \in A_x(\Omega)$ the energy functional has the form

$$E(m) = |\omega| \|\partial_x m\|_{L^2(\mathbb{R})}^2 + E_v(m) + E_s(m).$$

6 The representation of E_s in Fourier space

Next we find the representation of E_s in Fourier space which will make it more transparent. Let \mathcal{S} denote the Schwartz class. Recall some properties of Fourier transform:

1. $\widehat{\left(\frac{\partial f}{\partial \xi_j}\right)} = i\xi_j \hat{f}$ for all $f \in \mathcal{S}$
2. (Parseval's equality) $\int_{\mathbb{R}^n} |f|^2 d\xi = \int_{\mathbb{R}^n} |\hat{f}|^2 d\xi$ for all $f \in \mathcal{S}$
3. $\int_{\mathbb{R}^n} |\nabla f|^2 d\xi = \int_{\mathbb{R}^n} \frac{|\Delta f|^2}{|\xi|^2} d\xi$ for all $f \in \mathcal{S}$ and $n \geq 3$.

By the density argument the first equality is also valid for all $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\frac{\partial f}{\partial \xi_j} \in L^2(\mathbb{R}^n)$. The third equality is valid if $\nabla f \in L^2(\mathbb{R}^n)$ and $\frac{\Delta f}{|\xi|} \in L^2(\mathbb{R}^n)$ even when Δf is a distribution. For a given surface $\Gamma \subset \mathbb{R}^3$ we denote the Hausdorff H^2 measure restricted to Γ by δ_Γ . Let the point $(0, 0)$ be the center of symmetry of ω and let the parametrization

$$\begin{cases} y = y(t), & t \in [0, 2] \\ z = z(t), & t \in [0, 2] \end{cases}$$

of $\partial\omega$ be chosen by symmetry so that $y(t+1) = -y(t)$, $z(t+1) = -z(t)$. Denote by $\nu(t)$ the outward unit normal to $\partial\omega$ at $(y(t), z(t))$.

Theorem 6.1. For every $m \in M_x(\Omega)$ with a finite energy,

$$E_s(m) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{1}{|k|^2} \left\{ |a|^2 |\hat{m}_y(k_1)|^2 + |b|^2 |\hat{m}_z(k_1)|^2 + \bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)} + \overline{\hat{m}_y(k_1)}\hat{m}_z(k_1)) \right\} dk,$$

where

$$\begin{aligned} a(k_2, k_3, \omega) &= -2i \int_0^1 z'(t) \sin(k_2 y(t) + k_3 z(t)) dt, \\ b(k_2, k_3, \omega) &= 2i \int_0^1 y'(t) \sin(k_2 y(t) + k_3 z(t)) dt. \end{aligned}$$

Proof. We have for any $k \in \mathbb{R}^3$

$$\widehat{s \circ \delta_{\partial\omega}}(k) = \frac{1}{2\pi\sqrt{2\pi}} \int_{\mathbb{R}^3} e^{-i\xi k} (s \circ \delta_{\partial\omega})(\xi) d\xi.$$

It is clear that

$$\begin{aligned} \int_{\mathbb{R}^3} e^{-i\xi k} (s \circ \delta_{\partial\omega})(\xi) d\xi &= \int_{\mathbb{R}} \int_{\partial\omega} e^{-i(k_2 y + k_3 z)} m(x) \nu(y, z) dy dz \cdot e^{-ik_1 x} dx \\ &= \sqrt{2\pi} \hat{m}_y(k_1) \int_{\partial\omega} e^{-i(k_2 y + k_3 z)} \nu(y, z) dy dz \\ &\quad + \sqrt{2\pi} \hat{m}_z(k_1) \int_{\partial\omega} e^{-i(k_2 y + k_3 z)} \nu(y, z) dy dz \\ &= \sqrt{2\pi} \hat{m}_y(k_1) \int_0^2 z'(t) e^{-i(k_2 y(t) + k_3 z(t))} dt \\ &\quad - \sqrt{2\pi} \hat{m}_z(k_1) \int_0^2 y'(t) e^{-i(k_2 y(t) + k_3 z(t))} dt. \end{aligned}$$

For convenience we investigate the two parameters a and b as follows:

$$a(k_2, k_3, \omega) = \int_0^2 z'(t) e^{-i(k_2 y(t) + k_3 z(t))} dt,$$

$$b(k_2, k_3, \omega) = - \int_0^2 y'(t) e^{-i(k_2 y(t) + k_3 z(t))} dt.$$

Note that since the curve $\partial\omega$ is closed

$$k_3 a - k_2 b = \int_0^2 (k_3 z'(t) + k_2 y'(t)) e^{-i(k_2 y(t) + k_3 z(t))} dt = 0, \quad (8)$$

$$\begin{aligned} a(k_2, k_3, \omega) &= \int_0^1 z'(t) e^{-i(k_2 y(t) + k_3 z(t))} dt - \int_0^1 z'(t) e^{i(k_2 y(t) + k_3 z(t))} dt \\ &= -2i \int_0^1 z'(t) \sin(k_2 y(t) + k_3 z(t)) dt. \end{aligned}$$

Similarly we have

$$b(k_2, k_3, \omega) = 2i \int_0^1 y'(t) \sin(k_2 y(t) + k_3 z(t)) dt.$$

From the distributional identity

$$\Delta u_s = -s \circ \delta_{\partial\omega}$$

we obtain for the Fourier transform of Δu_s

$$\begin{aligned} |\widehat{\Delta u_s}(k)|^2 &= \frac{1}{4\pi^2} |a\hat{m}_y(k_1) + b\hat{m}_z(k_1)|^2 \\ &= \frac{1}{4\pi^2} (|a|^2 |\hat{m}_y(k_1)|^2 + |b|^2 |\hat{m}_z(k_1)|^2 \\ &\quad + \bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)} + \overline{\hat{m}_y(k_1)}\hat{m}_z(k_1)). \end{aligned}$$

Finally utilizing the third property of Fourier transform we obtain,

$$\begin{aligned} E_s(m) &= \int_{\mathbb{R}^3} |\nabla u_s(k)|^2 dk \\ &= \int_{\mathbb{R}^3} \frac{|\widehat{\Delta u_s}(k)|^2}{|k|^2} dk \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{1}{|k|^2} \left\{ |a|^2 |\hat{m}_y(k_1)|^2 + |b|^2 |\hat{m}_z(k_1)|^2 \right. \\ &\quad \left. + \bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)} + \overline{\hat{m}_y(k_1)}\hat{m}_z(k_1)) \right\} dk. \end{aligned}$$

□

7 An approximation for the magnetostatic energy

Next we prove that for sufficiently small diameters d the magnetostatic energy can be approximated by a quadratic form.

Theorem 7.1. *Determine*

$$I_\omega^1(k_1) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{|k|^2} dk_2 dk_3, \quad I_\omega^2(k_1) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{|b|^2}{|k|^2} dk_2 dk_3,$$

$$I_\omega^3(k_1) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{\bar{a}b}{|k|^2} dk_2 dk_3.$$

Then $I_\omega^1(0)$, $I_\omega^2(0)$ and $I_\omega^3(0)$ are given by

$$I_\omega^1(0) = \pi \int_{[0,1]^2} \ln \frac{(y(t) - y(t_1))^2 + (z(t) - z(t_1))^2}{(y(t) + y(t_1))^2 + (z(t) + z(t_1))^2} z'(t) z'(t_1) dt dt_1,$$

$$I_\omega^2(0) = \pi \int_{[0,1]^2} \ln \frac{(y(t) - y(t_1))^2 + (z(t) - z(t_1))^2}{(y(t) + y(t_1))^2 + (z(t) + z(t_1))^2} y'(t) y'(t_1) dt dt_1,$$

$$I_\omega^3(0) = \pi \int_{[0,1]^2} \ln \frac{(y(t) - y(t_1))^2 + (z(t) - z(t_1))^2}{(y(t) + y(t_1))^2 + (z(t) + z(t_1))^2} y'(t) z'(t_1) dt dt_1.$$

Proof. For convenience denote $y = y(t)$, $y_1 = y(t_1)$, $z = z(t)$ and $z_1 = z(t_1)$. We have that

$$\begin{aligned} |a|^2 &= 4 \left(\int_0^1 z'(t) \sin(k_2 y(t) + k_3 z(t)) dt \right)^2 \\ &= 4 \int_0^1 \int_0^1 z'(t) z'(t_1) \sin(k_2 y(t) + k_3 z(t)) \sin(k_2 y(t_1) + k_3 z(t_1)) dt dt_1 \\ &= 2 \int_0^1 \int_0^1 z' z_1' (\cos(k_2(y - y_1) + k_3(z - z_1)) - \cos(k_2(y + y_1) + k_3(z + z_1))) dt dt_1. \end{aligned}$$

We have as well

$$I_\omega^1(0) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{|k|^2} dk_2 dk_3 = 2 \int_0^1 \int_0^1 z' z_1' I_1^* dt dt_1,$$

where

$$I_1^* = \int_0^{+\infty} \int_{\mathbb{R}} \frac{\cos(k_2(y - y_1) + k_3(z - z_1)) - \cos(k_2(y + y_1) + k_3(z + z_1))}{k_2^2 + k_3^2} dk_2 dk_3.$$

Make the following notation:

$$p = |y - y_1|, \quad q = (z - z_1) \text{sign}(y - y_1), \quad r = |y + y_1|, \quad s = (z + z_1) \text{sign}(y + y_1).$$

Taking into account (27) and (28) we obtain

$$I_1^* = \pi \int_0^{+\infty} \frac{1}{k_3} (e^{-pk_3} \cos qk_3 - e^{-rk_3} \cos sk_3) dk_3 = \frac{\pi}{2} \ln \frac{p^2 + q^2}{r^2 + s^2},$$

thus the first formula is proven. The proofs of the other two formulas are analogues. \square

Theorem 7.2. Assume that $l > 0$. Then for any $k_1 \in [-l, l]$ the following bounds hold:

$$\begin{aligned} |I_\omega^1(0) - I_\omega^1(k_1)| &\leq 8\pi(\pi + 3)ld(\text{per}(\partial\omega))^2, \\ |I_\omega^2(0) - I_\omega^2(k_1)| &\leq 8\pi(\pi + 3)ld(\text{per}(\partial\omega))^2, \\ |I_\omega^3(0) - I_\omega^3(k_1)| &\leq 60(ld + 4(ld)^{\frac{4}{3}} + 3(ld)^{\frac{1}{3}})(\text{per}(\partial\omega))^2. \end{aligned}$$

Proof. First we estimate the difference $|I_\omega^1(k_1) - I_\omega^1(0)|$, the estimate for $I_\omega^2(k_1)$ is straightforward. The validity of the inequality $I_\omega^1(k_1) \leq I_\omega^1(0)$ for any $k_1 \in \mathbb{R}$ is evident. Note that if $k_1 \in [-l, l]$ then

$$I_\omega^1(k_1) \geq \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{k_2^2 + (k_3 + l)^2} dk_2 dk_3$$

thus we obtain

$$\begin{aligned} I_\omega^1(0) - I_\omega^1(k_1) &\leq \int_0^{+\infty} \int_{\mathbb{R}} |a|^2 \left(\frac{1}{k_2^2 + k_3^2} - \frac{1}{k_2^2 + (k_3 + l)^2} \right) dk_2 dk_3 \\ &\leq 2 \int_0^1 \int_0^1 |z' z'_1| (|J_1^1| + |J_2^1| + |J_3^1|) dt dt_1, \end{aligned}$$

where

$$\begin{aligned} J_1^1 &= \int_0^l \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3) - \cos(rk_2 + sk_3)}{k_2^2 + k_3^2} dk_2 dk_3, \\ J_2^1 &= \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3) - \cos(pk_2 + q(k_3 - l))}{k_2^2 + k_3^2} dk_2 dk_3, \\ J_3^1 &= \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(rk_2 + sk_3) - \cos(rk_2 + s(k_3 - l))}{k_2^2 + k_3^2} dk_2 dk_3. \end{aligned}$$

We have by (27),

$$\begin{aligned} |J_1^1| &= \left| 2 \int_0^l \int_0^{+\infty} \frac{\cos pk_2 \cos qk_3 - \cos rk_2 \cos sk_3}{k_2^2 + k_3^2} dk_2 dk_3 \right| \\ &= \pi \left| \int_0^l \frac{e^{-pk_3} \cos qk_3 - e^{-rk_3} \cos sk_3}{k_3} dk_3 \right| \\ &\leq \pi \int_0^l \frac{e^{-pk_3} |\cos qk_3 - \cos sk_3|}{k_3} dk_3 + \pi \int_0^l \frac{|\cos sk_3 (e^{-pk_3} - e^{-rk_3})|}{k_3} dk_3 \\ &\leq 2\pi \int_0^l \frac{|\sin \frac{q+s}{2} k_3 \sin \frac{q-s}{2} k_3|}{k_3} dk_3 + \pi \int_0^l \frac{1}{k_3} \left| \int_p^r \frac{d}{dt} (e^{-k_3 t}) dt \right| dk_3 \\ &\leq \pi l |q - s| + \pi |p - r| \int_0^l \max(e^{-pk_3}, e^{-rk_3}) dk_3 \\ &\leq 4\pi dl. \end{aligned}$$

By virtue of Lemma A.4 we have

$$\begin{aligned} |J_2^1| &\leq (1 - \cos ql) \left| \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3)}{k_2^2 + k_3^2} dk_2 dk_3 \right| \\ &\quad + |\sin ql| \left| \int_l^{+\infty} \int_{\mathbb{R}} \frac{\sin(pk_2 + qk_3)}{k_2^2 + k_3^2} dk_2 dk_3 \right| \\ &= 4 \sin^2 \frac{ql}{2} \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos pk_2 \cos qk_3}{k_2^2 + k_3^2} dk_2 dk_3 \right| \\ &\quad + 2 |\sin ql| \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos pk_2 \sin qk_3}{k_2^2 + k_3^2} dk_2 dk_3 \right| \\ &\leq (ql)^2 \cdot \frac{2\pi}{ql} + \frac{\pi^2}{2} 2ql \\ &\leq (4\pi + 2\pi^2) dl. \end{aligned}$$

Similarly $|J_3^1| \leq (4\pi + 2\pi^2) dl$. Concluding we obtain

$$|J_1^1| + |J_2^1| + |J_3^1| \leq 4\pi(\pi + 3)dl,$$

thus

$$\begin{aligned} I_\omega^1(0) - I_\omega^1(k_1) &\leq 8\pi(\pi + 3)dl \left(\int_0^1 |z'(t)| dt \right)^2 \\ &\leq 8\pi(\pi + 3)dl(\text{per}(\omega))^2. \end{aligned}$$

Analogously we have

$$I_\omega^2(0) - I_\omega^2(k_1) \leq 8\pi(\pi + 3)dl(\text{per}(\omega))^2.$$

To estimate $|I_\omega^3(0) - I_\omega^3(k_1)|$ we recall that $b = \frac{k_3}{k_2}a$, thus

$$I_\omega^3(k_1) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{k_3|a|^2}{k_2|k|^2} dk_2 dk_3.$$

Note that the integrand is positive if $k_2 > 0$ and negative if $k_2 < 0$, therefore

$$\begin{aligned} |I_\omega^3(0) - I_\omega^3(k_1)| &\leq \int_0^{+\infty} \int_0^{+\infty} \frac{k_3|a|^2}{k_2} \left(\frac{1}{k_2^2 + k_3^2} - \frac{1}{|k|^2} \right) dk_2 dk_3 \\ &\quad + \int_0^{+\infty} \int_{-\infty}^0 \frac{k_3|a|^2}{k_2} \left(\frac{1}{|k|^2} - \frac{1}{k_2^2 + k_3^2} \right) dk_2 dk_3 \\ &\leq \int_0^{+\infty} \int_0^{+\infty} \bar{a}b \left(\frac{1}{k_2^2 + k_3^2} - \frac{1}{k_2^2 + (k_3 + l)^2} \right) dk_2 dk_3 \\ &\quad + \int_0^{+\infty} \int_{-\infty}^0 \bar{a}b \left(\frac{1}{k_2^2 + (k_3 + l)^2} - \frac{1}{k_2^2 + k_3^2} \right) dk_2 dk_3. \end{aligned}$$

We have that

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \bar{a}b \left(\frac{1}{k_2^2 + k_3^2} - \frac{1}{k_2^2 + (k_3 + l)^2} \right) dk_2 dk_3 \\ &\leq 2 \int_0^1 \int_0^1 |z'y'_1| (|J_1^3| + |J_2^3| + |J_3^3|) dt dt_1, \end{aligned}$$

where

$$\begin{aligned} J_1^3 &= \int_0^l \int_0^{+\infty} \frac{\cos(pk_2 + qk_3) - \cos(rk_2 + sk_3)}{k_2^2 + k_3^2} dk_2 dk_3, \\ J_2^3 &= \int_l^{+\infty} \int_0^{+\infty} \frac{\cos(pk_2 + qk_3) - \cos(pk_2 + q(k_3 - l))}{k_2^2 + k_3^2} dk_2 dk_3, \\ J_3^3 &= \int_l^{+\infty} \int_0^{+\infty} \frac{\cos(rk_2 + sk_3) - \cos(rk_2 + s(k_3 - l))}{k_2^2 + k_3^2} dk_2 dk_3. \end{aligned}$$

Utilizing now Lemmas A2 and A3, and the estimate for J_1^1 we get

$$\begin{aligned}
|J_1^3| &\leq \frac{|J_1^1|}{2} + \left| \int_0^l I(p, k_3) \sin qk_3 \, dk_3 \right| + \left| \int_0^l I(r, k_3) \sin sk_3 \, dk_3 \right| \\
&\leq 2\pi ld + 7p^{\frac{1}{3}} \int_0^l \frac{qk_3}{k_3^{\frac{2}{3}}} \, dk_3 + 7r^{\frac{1}{3}} \int_0^l \frac{sk_3}{k_3^{\frac{2}{3}}} \, dk_3 \\
&< 2\pi ld + 30(ld)^{\frac{4}{3}}.
\end{aligned}$$

Utilizing Lemma A4 we get

$$\begin{aligned}
|J_2^3| &\leq 2 \sin^2 \frac{ql}{2} \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos(pk_2 + qk_3)}{k_2^2 + k_3^2} \, dk_2 \, dk_3 \right| \\
&\quad + |\sin ql| \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\sin(pk_2 + qk_3)}{k_2^2 + k_3^2} \, dk_2 \, dk_3 \right| \\
&\leq \frac{(ql)^2}{2} \left(\frac{2\pi}{ql} + \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}} \right) + ql \left(\frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}} + \frac{\pi^2}{4} \right) \\
&< 10(3ld + 4(ld)^{\frac{1}{3}} + 4(ld)^{\frac{4}{3}}).
\end{aligned}$$

Similarly we have

$$|J_3^3| < 10(3ld + 4(ld)^{\frac{1}{3}} + 4(ld)^{\frac{4}{3}}).$$

Concluding we obtain

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \bar{a}b \left(\frac{1}{k_2^2 + k_3^2} - \frac{1}{k_2^2 + (k_3 + l)^2} \right) \, dk_2 \, dk_3 \\
&< 20(7ld + 8(ld)^{\frac{1}{3}} + 11(ld)^{\frac{4}{3}})(\text{per}(\partial\omega))^2.
\end{aligned}$$

The same estimate for

$$\int_0^{+\infty} \int_{-\infty}^0 \bar{a}b \left(\frac{1}{k_2^2 + (k_3 + l)^2} - \frac{1}{k_2^2 + k_3^2} \right) \, dk_2 \, dk_3$$

is straightforward. For I_ω^3 we get

$$|I_\omega^3(0) - I_\omega^3(k_1)| \leq 40(7ld + 8(ld)^{\frac{1}{3}} + 11(ld)^{\frac{4}{3}})(\text{per}(\partial\omega))^2 \quad \text{for all } k_1 \in [-l, l].$$

□

Corollary 7.3. Denote $u = d^{\frac{1}{6}}(\text{per}(\omega))^2$. Then for sufficiently small d and for any $k_1 \in \left[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right]$ we have

$$\begin{aligned}
|I_\omega^1(0) - I_\omega^1(k_1)| &\leq u, \\
|I_\omega^2(0) - I_\omega^2(k_1)| &\leq u, \\
|I_\omega^3(0) - I_\omega^3(k_1)| &\leq 400u.
\end{aligned}$$

Now we are in the situation to prove an approximation theorem for the magnetostatic energy. For convenience we denote $A_\omega = I_\omega^1(0)$, $B_\omega = I_\omega^2(0)$, $C_\omega = I_\omega^3(0)$. By virtue of Theorem 7.1 the parameters A_ω , B_ω and C_ω depend homogeneously on the diameter of ω with exponent 2.

Theorem 7.4. *Assume $m \in M_x(\Omega)$ and $m_y, m_z \in L^2(\mathbb{R})$. Define*

$$E_s^*(m) = \frac{1}{2\pi^2} \int_{\mathbb{R}} (A_\omega |m_y(x)|^2 + B_\omega |m_z(x)|^2 + C_\omega (\hat{m}_y(x) \overline{\hat{m}_z(x)} + \overline{\hat{m}_y(x)} \hat{m}_z(x))) dx.$$

For sufficiently small d the following inequality holds:

$$|E_s(m) - E_s^*(m)| \leq 23u \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) dx + \frac{(A_\omega + B_\omega)E_{ex}(m)}{\pi^2 |\omega| d}.$$

Proof. For any $l > 0$ we have that

$$\begin{aligned} E_{ex}(m) &= |\omega| d^2 \int_{\mathbb{R}} |\partial_x m(x)|^2 dx \\ &\geq |\omega| d^2 \int_{\mathbb{R}} (|\partial_x m_y(x)|^2 + |\partial_x m_z(x)|^2) dx \\ &= |\omega| d^2 \int_{\mathbb{R}} (|\widehat{\partial_x m_y}(x)|^2 + |\widehat{\partial_x m_z}(x)|^2) dx \\ &= |\omega| d^2 \int_{\mathbb{R}} |x|^2 (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dx \\ &\geq |\omega| d^2 l^2 \int_{\mathbb{R} \setminus [-l, l]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dx, \end{aligned}$$

which implies for $l = \frac{1}{\sqrt{d}}$,

$$\int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dx \leq \frac{E_{ex}(m)}{c_\omega d}. \quad (9)$$

It is clear that for any $k_1 \in \mathbb{R}$,

$$\begin{aligned} |I_\omega^3(k_1)| &\leq \int_0^{+\infty} \int_{\mathbb{R}} \frac{|\bar{a}b|}{|k|^2} dk_2 dk_3 \\ &\leq \left(\int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{|k|^2} dk_2 dk_3 \cdot \int_0^{+\infty} \int_{\mathbb{R}} \frac{|b|^2}{|k|^2} dk_2 dk_3 \right)^{\frac{1}{2}} \\ &= (I_\omega^1(k_1) \cdot I_\omega^2(k_1))^{\frac{1}{2}} \leq (A_\omega B_\omega)^{\frac{1}{2}} \leq \frac{A_\omega + B_\omega}{2}, \end{aligned}$$

thus

$$|I_\omega^3(k_1)| \leq \frac{A_\omega + B_\omega}{2} \quad (10)$$

Utilizing Corollary 7.3 and inequalities (9), (10) we obtain

$$\begin{aligned}
|E_s(m) - E_s^*(m)| &\leq \frac{1}{2\pi^2} \int_{\mathbb{R}} |A_\omega - I_\omega^1(x)| |m_y(x)|^2 dx \\
&\quad + \frac{1}{2\pi^2} \int_{\mathbb{R}} |B_\omega - I_\omega^2(x)| |m_z(x)|^2 dx \\
&\quad + \frac{1}{2\pi^2} \int_{\mathbb{R}} |C_\omega - I_\omega^3(x)| |\hat{m}_y(x) \overline{\hat{m}_z(x)} + \overline{\hat{m}_y(x)} \hat{m}_z(x)| dx \\
&\leq \frac{u}{2\pi^2} \int_{-\frac{1}{\sqrt{d}}}^{\frac{1}{\sqrt{d}}} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dx \\
&\quad + \frac{A_\omega}{2\pi^2} \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]} |\hat{m}_y(x)|^2 dx + \frac{B_\omega}{2\pi^2} \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]} |\hat{m}_z(x)|^2 dx \\
&\quad + \frac{400u}{2\pi^2} \int_{-\frac{1}{\sqrt{d}}}^{\frac{1}{\sqrt{d}}} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dx \\
&\quad + \frac{A_\omega + B_\omega}{2\pi^2} \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2) dk_1 \\
&\leq 23u \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) dx + \frac{(A_\omega + B_\omega)E_{ex}(m)}{\pi^2|\omega|d}.
\end{aligned}$$

□

Lemma 7.5. For any numbers $0 < s \leq r$ denote $R(s, r) = [-s, s] \times [-r, r]$. Then for all points $(y_1, z_1) \in R(s, r)$ the following bound holds:

$$I = \int_{R(s, r)} \frac{dy dz}{\sqrt{(y - y_1)^2 + (z - z_1)^2}} < 10s \left(1 + \ln \frac{r}{s}\right).$$

Proof. It is clear that

$$\begin{aligned}
I &\leq \int_{R(2s, 2r)} \frac{dy dz}{\sqrt{y^2 + z^2}} = \int_{R(2s, 2s)} \frac{dy dz}{\sqrt{y^2 + z^2}} + \int_{R(2s, 2r) \setminus R(2s, 2s)} \frac{dy dz}{\sqrt{y^2 + z^2}} \\
&\leq \frac{1}{4} \int_{D_{4\sqrt{2}s}(0)} \frac{dy dz}{\sqrt{y^2 + z^2}} + 8s \int_{2s}^{2r} \frac{dy}{y} \\
&= 2\sqrt{2}\pi d + 8s \ln \frac{r}{s} \\
&< 10s \left(1 + \ln \frac{r}{s}\right).
\end{aligned}$$

□

Lemma 7.6. For any $m \in A_x(\Omega)$ there exists a constant M_m depending only on m such that

$$E_v(m) \leq M_m d^3 (1 + d).$$

Proof. By density argument equality (4) holds for $\varphi = u_v$, thus utilizing Theorem 5.1 we obtain

$$E_v(m) = \int_{\mathbb{R}^3} |\nabla u_v|^2 = \int_{\Omega} v \cdot u_v = \int_{\Omega} \int_{\Omega} \Gamma(\xi - \xi_1) v(\xi) v(\xi_1) d\xi d\xi_1.$$

We have that $m \in A_x(\Omega)$ so $v(x, y, z) = \partial_x m_x(x)$ thus

$$E_v(m) = \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} \frac{\partial_x m_x(x) \partial_x m_{x_1}(x_1)}{|\xi - \xi_1|} d\xi d\xi_1$$

where $\xi = (x, y, z)$ and $\xi_1 = (x_1, y_1, z_1)$.

It is clear that

$$\begin{aligned} \int_{\mathbb{R}} \frac{\partial_x m_x(x)}{|\xi - \xi_1|} dx &= \int_{-\infty}^0 \frac{dm^*(x)}{|\xi - \xi_1|} + \int_0^{+\infty} \frac{dm^*(x)}{|\xi - \xi_1|} \\ &= \frac{2}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}} - \int_{\mathbb{R}} \frac{(x - x_1)m^*(x)}{|\xi - \xi_1|^3} dx, \end{aligned}$$

hence taking into account the fact that ω can be placed in a square with sides parallel to the y and z axis and lengths d we obtain for the energy

$$E_v(m) \leq \frac{1}{2\pi} I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{R(d,d)} \int_{\Omega} \frac{|\partial_x m_x(x_1)|}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}} d\xi_1 dy dz, \\ I_2 &= \int_{\Omega} \int_{\Omega} \frac{|\partial_x m_x(x_1)m^*(x)|}{|\xi - \xi_1|^2} d\xi d\xi_1. \end{aligned}$$

We have

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\partial_x m_x(x_1)|}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}} dx_1 \\ \leq \frac{1}{2} \int_{\mathbb{R}} \left(|\partial_x m_x(x_1)|^2 + \frac{1}{x_1^2 + (y - y_1)^2 + (z - z_1)^2} \right) dx_1 \\ = \frac{1}{2} \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2\sqrt{(y - y_1)^2 + (z - z_1)^2}}. \end{aligned}$$

Utilizing now Lemma 7.5 we get

$$\begin{aligned} I_1 &\leq \frac{4}{\pi} \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 d^4 + \frac{1}{4} \int_{R(d,d)} \int_{R(d,d)} \frac{1}{\sqrt{(y - y_1)^2 + (z - z_1)^2}} dy_1 dz_1 dy dz \\ &\leq \frac{4}{\pi} \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 d^4 + 10d^3. \end{aligned}$$

By making a change of variables $\xi_2 = \xi_1 - \xi$ and utilizing again Lemma 7.5 we get

$$\begin{aligned} I_2 &= \int_{\Omega} \int_{\mathbb{R} \times [-d-y, d-y] \times [-d-z, d-z]} \frac{|m^*(x)| \cdot |\partial_x m_x(x_2 + x)|}{|\xi_2|^2} d\xi_2 d\xi \\ &\leq \frac{1}{2} \int_{R(d,d)} \int_{\mathbb{R} \times [-d-y, d-y] \times [-d-z, d-z]} \int_{\mathbb{R}} \frac{|m^*(x)|^2 + |\partial_x m_x(x_2 + x)|^2}{|\xi_2|^2} dx d\xi_2 dy dz \\ &= 2d^2 \left(\|m^*\|_{L^2(\mathbb{R})}^2 + \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 \right) \int_{\mathbb{R} \times [-d-y, d-y] \times [-d-z, d-z]} \frac{d\xi_2}{|\xi_2|^2} \\ &= 2\pi d^2 \left(\|m^*\|_{L^2(\mathbb{R})}^2 + \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 \right) \int_{R(d,d)} \frac{1}{\sqrt{(y_1 - y)^2 + (z_1 - z)^2}} dy_1 dz_1 \\ &\leq 20\pi d^3 \left(\|m^*\|_{L^2(\mathbb{R})}^2 + \|\partial_x m_x\|_{L^2(\mathbb{R})}^2 \right). \end{aligned}$$

The summary of the estimates for I_1 and I_2 and Corollary 5.4 completes the proof. \square

8 Existence of minimizers

In the next step we prove a compactness lemma which will be crucial in both the existence and the Γ -convergence theorems.

Lemma 8.1. *Assume that the sequence of magnetizations $\{m^n\}$ satisfies $m^n: \Omega \rightarrow \mathbb{S}^2$ and $E(m^n) \leq C$ for some constant C . Then there exists a magnetization $m^0: \Omega \rightarrow \mathbb{S}^2$ such that for a subsequence of $\{m^n\}$ (not relabeled) the following statements hold:*

- (i) $\nabla m^n \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$
- (ii) $m^n \rightarrow m^0$ strongly in $L^2_{loc}(\Omega)$
- (iii) $E(m^0) \leq \liminf E(m^n)$.

Proof. Let u_n be the weak solution of $\Delta u = \operatorname{div} m^n$. We have that

$$\int_{\Omega} |\nabla m^n|^2 + \int_{\mathbb{R}^3} |\nabla u^n|^2 \leq C$$

thus $\{\nabla m^n\}$ and $\{\nabla u_n\}$ contain weakly convergent subsequences (not relabeled), i.e.,

$$\nabla m^n \rightharpoonup f \quad \text{weakly in} \quad L^2(\Omega) \quad (11)$$

and

$$\nabla u_n \rightharpoonup g \quad \text{weakly in} \quad L^2(\mathbb{R}^3) \quad (12)$$

for some $f \in L^2(\Omega)$ and $g \in L^2(\mathbb{R}^3)$. Since $|m^n| = 1$ in Ω we have that

$$m^n \in W^{1,2}([-N, N] \times \omega) \quad \text{for any} \quad N \in \mathbb{N}.$$

Taking into account the fact that the embedding

$$W^{1,2}([-N, N] \times \omega) \hookrightarrow L^2([-N, N] \times \omega)$$

is compact, one can extract a subsequence from the new subsequence $\{m^n\}$ (not relabeled) converging to some m^0 in $L^2([-N, N] \times \omega)$. Doing this for all natural values of N and applying a diagonal argument to the extracted subsequences we obtain a subsequence of $\{m^n\}$ (not relabeled) such that in addition to (11) and (12) we have $m^n \rightarrow m^0$ strongly in $L^2_{loc}(\Omega)$.

Applying a standard argument we can deduce that m^0 is weakly differentiable and $\nabla m^0 = f$. We extend m^0 outside Ω as zero. A standard argument then shows that

$$\int_{\mathbb{R}^3} m^0 \cdot \nabla \varphi \, d\xi = \int_{\mathbb{R}^3} g \cdot \nabla \varphi \, d\xi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3).$$

Since $g \in L^2(\mathbb{R}^3)$ the equation $\Delta u = \operatorname{div} g$ has a weak solution u_0 which is equivalent to

$$\int_{\mathbb{R}^3} g \cdot \nabla \varphi \, d\xi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \varphi \, d\xi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3),$$

thus

$$\int_{\mathbb{R}^3} m^0 \cdot \nabla \varphi \, d\xi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \varphi \, d\xi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3)$$

which means that u_0 is a weak solution of

$$\Delta u = \operatorname{div} m^0.$$

Since $g \in L^2(\mathbb{R}^3)$ we already know that

$$\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^2(\mathbb{R}^3)}.$$

Therefore by virtue of (11) and (12) we obtain

$$\begin{aligned} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} &\leq \|g\|_{L^2(\mathbb{R}^3)} \leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(\mathbb{R}^3)} \\ \|\nabla m^0\|_{L^2(\mathbb{R}^3)} &\leq \liminf_{n \rightarrow \infty} \|\nabla m^n\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

which yields

$$E(m^0) \leq \liminf_{n \rightarrow \infty} E(m^n).$$

□

We are now in a situation to prove an existence theorem which is the analog of theorem (17) in [19] for C^2 cross sections.

Theorem 8.2 (Existence). *For every bounded, simply connected and C^2 domain ω there exist a minimizer of E is \tilde{A} .*

Proof. Let m^n be any minimizing sequence, i.e.,

$$\lim_{n \rightarrow \infty} E(m^n) = \inf_{m \in \tilde{A}(\Omega)} E(m) := E_{\min}(\Omega).$$

Since the sequence $\{E(m^n)\}$ is bounded we can extract a subsequence $\{m^n\}$ (not relabeled) that has a limit $m^0 \in W_{loc}^{1,2}(\Omega)$ in the sense mentioned in Lemma 8.1. If we could show that $m^0 \in \tilde{A}$ then m^0 would be the desired minimizer because of the fact that

$$E(m^0) \leq \liminf_{n \rightarrow \infty} E(m^n) = E_{\min}(\Omega).$$

But m^0 does not have to belong to \tilde{A} in general because the boundary conditions at $\pm\infty$ could be violated. To overcome this difficulty we construct a minimizing sequence so that its limit belongs to \tilde{A} . We choose any minimizing sequence $\{m^n\}$ as above with a limit m^0 in the sense mentioned in Lemma 8.1. The idea is to show that the desired minimizing sequence can be constructed by translating each magnetization m^n by a factor x_n in the x coordinate direction. First of all note that the minimization problem is invariant under translations in the first coordinate, that is if $m \in \tilde{A}$ then obviously $m_c(x, y, z) = m(x - c, y, z) \in \tilde{A}$ and $E(m_c) = E(m)$. Since $E(m^n) \rightarrow E_{\min}(\Omega)$ then $|E(m^n)| \leq M$ for some M and for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ consider the sets A_n , B_n and C_n defined in the following way:

$$\begin{aligned}
A_n &= \left\{ x \in \mathbb{R} : -1 \leq \bar{m}_x^n(x) < -\frac{1}{2} \right\} \\
B_n &= \left\{ x \in \mathbb{R} : -\frac{1}{2} < \bar{m}_x^n(x) < \frac{1}{2} \right\} \\
C_n &= \left\{ x \in \mathbb{R} : \frac{1}{2} < \bar{m}_x^n(x) \leq 1 \right\}
\end{aligned}$$

Since \bar{m}_x^n is continuous in \mathbb{R} then for all $n \in \mathbb{N}$, A_n , B_n and C_n are a finite or countable union of disjoint intervals. By virtue of Lemma 4.7 and Remark 4.8 one of the intervals in A_n has the form $(-\infty, a_n)$ and one of the intervals in C_n has the form $(c_n, +\infty)$ (note that \bar{m}_x^n is negative at $-\infty$ and positive at $+\infty$.) We distinguish two types of intervals in B_n . The interval $(a, b) \subset B_n$ is said to be of the first type if $|\bar{m}^n(a) - \bar{m}^n(b)| = 1$, and of the second type otherwise. By Lemma 4.7 the sum of the lengths of all intervals, as well as the number of the first type intervals in B_n is bounded by a number depending only on M and ω , i.e., a constant not depending on n . Suppose first that there are no second type intervals in B_n for all $n \in \mathbb{N}$. Let us paint all the points of A_n , B_n and C_n with respectively black, yellow and red color for all $n \in \mathbb{N}$. We call the increasing sequence $\{n_k\}$ with natural entries "good" if for every $k \in \mathbb{N}$ there exist two intervals $(a_1^k, a_2^k) \subset A_{n_k}$ and $(c_1^k, c_2^k) \subset C_{n_k}$ such that

$$a_2^k - a_1^k \rightarrow +\infty, \quad c_2^k - c_1^k \rightarrow +\infty, \quad 0 < c_1^k - a_2^k \leq C$$

for a constant C not depending on k . The endpoints a_1^k and c_2^k can also take values $-\infty$ and $+\infty$ respectively. If this is the case, the subsequence $\{m^{n_k}\}$ will also be called "good". We prove that for any minimizing sequence $\{m^n\} \subset \bar{A}(\Omega)$ there exists a "good" subsequence $\{n_k\}$. For every fixed n there are some black, yellow and white intervals between $(-\infty, a_n)$ and $(c_n, +\infty)$. Note that if the number of yellow intervals is less than s then the number of both black and red intervals are less than $s + 1$ because there is obviously at least one yellow interval between any two black and any two red intervals. Therefore the number of all intervals is less than $3s + 2$. Since n was arbitrary we get that the number of all the intervals in the n -th family of the constructed intervals is bounded by the same number S . Let us number both the red and the black intervals in any family of intervals. We prove the existence of a "good" subsequence by induction in S but we first reformulate the problem as follows: Suppose we are given a sequence of natural numbers S_n and a sequence of families of S_n disjoint intervals on the real line painted with black and red color for all $n \in \mathbb{N}$. Assume $S_n \leq S$ and the sum of the lengths of $S_n - 1$ gaps between the intervals of the n -th family is bounded by the same number M for all $n \in \mathbb{N}$. Assume furthermore that for any $n \in \mathbb{N}$ the far left placed interval is black and the far right placed interval is red and their lengths tend to $+\infty$ as n goes to infinity. Then there exists a subsequence $\{n_k\}$ and two intervals (a_1^k, a_2^k) and (c_1^k, c_2^k) in the n_k -th family such that (a_1^k, a_2^k) is black, (c_1^k, c_2^k) is red, and

$$a_2^k - a_1^k \rightarrow +\infty, \quad c_2^k - c_1^k \rightarrow +\infty, \quad 0 < c_1^k - a_2^k \leq M_1 \quad (13)$$

for a constant M_1 and all $k \in \mathbb{N}$. The case $S = 2$ is evident. Assume it is true for $S \leq N$ and let us prove it for $S = N + 1$. Since $S \geq 3$, in every family there are at least two intervals of the same color. Assume that for infinitely many indices n there are at least two black intervals in the n -th family. We consider now the subsequence of the families with such indices. We consider the far right placed black intervals for all such families. There are two possible cases:

Case 1. *For a subsequence their lengths tend to $+\infty$.*

In this case we can omit all the intervals placed on their left side which leads to a situation with less intervals in every family (in such a subsequence) fulfilling the requirements of the statement, so by induction the existence of a "good" subsequence is proven.

Case 2. *Their lengths are bounded by the same number M_2 .*

In this case we can omit this intervals and this will lead us to a situation with less intervals in any family fulfilling the requirements of the statements so by the induction the existence of a "good" subsequence is proven

Let us get now back to our situation. If we omit all the yellow intervals from the real line for all $n \in \mathbb{N}$ then the families of the black and the red intervals fulfill the requirements of the statement proven above, thus the existence of a "good" sequence is proven. Take the two intervals $[a_1^k, a_2^k]$ and $[c_1^k, c_2^k]$ for all $k \in \mathbb{N}$ and denote the "good" sequence of the magnetizations again by $\{m^k\}$ which will also be a minimizing sequence. We transfer the origin of the real line to the point a_2^k for any m^k and denote

$$m_{good}^k(x, y, z) = m^k(x + a_2^k, y, z).$$

As we already know $\{m_{good}^k\}$ is a minimizing sequence and furthermore if we put $a_3^k = a_2^k - a_1^k$, $c_3^k = c_1^k - a_2^k$ and $c_4^k = c_2^k - a_2^k$ then

$$\bar{m}_{good}^k(x) \leq -\frac{1}{2} \text{ for } x \in [-a_3^k, 0] \text{ and } \bar{m}_{good}^k(x) \geq \frac{1}{2} \text{ for } x \in [c_3^k, c_4^k],$$

where

$$a_3^k \rightarrow +\infty, \quad c_4^k - c_3^k \rightarrow +\infty, \quad 0 < c_3^k < M_3.$$

By Lemma one can extract a subsequence from $\{m_{good}^k\}$ (not relabeled) with a limit $m^0 \in A(\Omega)$. Let us now prove that actually $m^0 \in \tilde{A}(\Omega)$. Recall that for any magnetization m the inclusions $m \pm \vec{e}_x \in L^2(\Omega_+)$, $m \pm \vec{e}_x \in L^2(\Omega_-)$ are equivalent to $1 \pm \bar{m}_x \in L^1(0, +\infty)$, $1 \pm \bar{m}_x \in L^1(-\infty, 0)$ respectively. Since $m^0 \in A(\Omega)$ according to Theorem 5.2 two of the four statements must hold: $1 \pm \bar{m}_x^0 \in L^1(0, +\infty)$, $1 \pm \bar{m}_x^0 \in L^1(-\infty, 0)$. We have for any fixed $R > 0$,

$$\begin{aligned} \int_{-R}^R |\bar{m}_x^0 - \bar{m}_{good,x}^k| dx &= \frac{1}{|\omega|} \int_{-R}^R \left| \int_{\omega} (m_x^0 - m_{good,x}^k) dy dz \right| dx \\ &\leq \frac{1}{|\omega|} \int_{-R}^R \int_{\omega} |m_x^0 - m_{good,x}^k| dy dz dx \\ &\leq \frac{1}{|\omega|} \left(2R|\omega| \cdot \int_{[-R,R] \times \omega} |m_x^0 - m_{good,x}^k|^2 d\xi \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2R}{|\omega|}} \cdot \|m_x^0 - m_{good,x}^k\|_{L^2([-R,R] \times \omega)} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ because of the strong convergence $m_{good}^k \rightarrow m^0$ in $L_{loc}^2(\Omega)$. Therefore a subsequence of $\{\bar{m}_{x,good}^k(x)\}$ converges pointwise to $\bar{m}_x^0(x)$ almost everywhere in $[-R, R]$. Giving R all natural values and applying diagonal argument we obtain that a subsequence of $\{\bar{m}_{x,good}^k(x)\}$ converges pointwise to $\bar{m}_x^0(x)$ almost everywhere in \mathbb{R} , therefore $\bar{m}_x^0(x) \leq -\frac{1}{2}$ a.e. in $(-\infty, 0)$ and $\bar{m}_x^0(x) \geq \frac{1}{2}$ a.e. in $[M_3, +\infty)$ which itself yields $1 - \bar{m}_x^0$ and $1 + \bar{m}_x^0$ can not belong to $L^1(-\infty, 0)$ and $L^1(0, +\infty)$ respectively, therefore $1 + \bar{m}_x^0 \in L^1(-\infty, 0)$ and $1 - \bar{m}_x^0 \in L^1(0, +\infty)$, thus $m^0 \in \tilde{A}(\Omega)$. The theorem is proven for the case when there is no second type yellow interval. Assume now that there are such intervals. Throwing away all the second type yellow intervals from the real line we can regard the rest of the real line as a real line without gaps simply by shifting all the intervals to the left hand side such that after that operation no overlap occurs and there is no gap left. To be more precise, we shift each interval to the left hand side by a factor equal to the sum of the lengths of the gaps between that interval and $-\infty$. During that operation we unify the black and red intervals with the neighboring intervals of the same color but we regard the possible neighboring first type yellow intervals as separate. We get a situation like above and therefore we can prove the existence of a "good" subsequence. It is easy to show that since that sum of the lengths of the second type yellow intervals in each family is bounded by the same constant then the in Lemma 8.1 described limit of the obtained "good" subsequence will belong to $\tilde{A}(\Omega)$ and hence will be an energy minimizer. \square

9 The convergence of the energies

Fix a cylindrical domain $\Omega = \mathbb{R} \times \omega \in \mathbb{R}^3$, where ω is a bounded, centrally symmetric and simply-connected domain with a C^2 boundary. Assume that the sequence $\{d_n\}$ satisfies $d_n \rightarrow 0$. Consider the sequence of homothetic domains Ω_n , where $\Omega_n = d_n \cdot \Omega$. By considering then minimization problem (1) in each domain Ω_n we get a sequence of minimization problems. We will rescale the magnetizations in the y and z directions as follows: $\acute{m}^n(x, y, z) = m^n(x, d_n y, d_n z)$ for $m^n(x, y, z) \in \Omega_n$. It is clear that $\acute{m}^n: \Omega \rightarrow \mathbb{S}^2$. Instead of minimization problem (1) we consider the problem

$$\inf_{m \in \tilde{A}(\Omega_n)} \frac{E(m)}{d_n^2}. \quad (14)$$

Denote $\acute{E}_n(\acute{m}) = \frac{E_n(m)}{d_n^2}$, where the superscript n indicates that the energy is regarded in Ω_n . It turns out that for any choice of the sequence $\{d_n\}$ the sequence of minimization problems (14) Γ -converges to a one dimensional problem that depends only on the domain Ω . It is clear that the rescaled energy functional will be

$$\acute{E}_n(\acute{m}) = \int_{\Omega} \left(|\partial_x \acute{m}(\xi)|^2 + \frac{1}{d_n^2} |\partial_y \acute{m}(\xi)|^2 + \frac{1}{d_n^2} |\partial_z \acute{m}(\xi)|^2 \right) d\xi + \frac{1}{d_n^2} E_{mag}^n(m).$$

The reduced variational problem energy functional is give by

$$E_0(m) = |\omega| \int_{\mathbb{R}} |\partial_x m|^2 dx + \frac{a_{\omega}}{2\pi^2} \int_{\mathbb{R}} (|m_y|^2 + |m_z|^2) dx$$

$$+\frac{C_\omega}{2\pi^2 t_\omega} \int_{\mathbb{R}} |t_\omega m_y + m_z|^2 dx, \quad \text{if } m = m(x),$$

$$E_0(m) = +\infty, \quad \text{otherwise}$$

where the numbers a_ω and t_ω are defined as follows:

$$t_\omega = \frac{A_\omega - B_\omega + \sqrt{(A_\omega - B_\omega)^2 + 4C_\omega^2}}{2C_\omega}$$

and

$$a_\omega = A_\omega - C_\omega t_\omega.$$

We will show later that $a_\omega, t_\omega > 0$.

The admissible set for the reduced variational problem is $\tilde{A}(\Omega)$, thus the reduced minimization problem is

$$\inf_{m \in \tilde{A}(\Omega)} E_0(m). \quad (15)$$

Define

$$X(\Omega) = \{m: \Omega \rightarrow \mathbb{R}^3 : \nabla m \in L^2(\Omega), m_y, m_z \in L_{loc}^2(\Omega)\}.$$

Now we define the notion of convergence of the magnetizations we are going to use for the Γ -convergence of the energies.

Definition 9.1. *The sequence $\{m^n\} \subset X(\Omega)$ is said to converge to $m \in X(\Omega)$ as n goes to infinity if,*

- (i) $\nabla m^n \rightharpoonup \nabla m$ weakly in $L^2(\Omega)$
- (ii) $m^n \rightarrow m$ strongly in $L_{loc}^2(\Omega)$

Theorem 9.2 (Γ -convergence). *The sequence of rescaled variational problems Γ -converge to the reduced variational problem. This amounts to the following:*

- (i) (**Lower semicontinuity**) *If a sequence of rescaled magnetizations $\{\acute{m}^n\}$ with $\acute{m}^n \in A(\Omega)$ converges to some $m^0 \in X(\Omega)$, then*

$$E_0(m^0) \leq \liminf_{n \rightarrow \infty} \acute{E}_n(\acute{m}^n)$$

- (ii) (**Construction**) *For every $m^0 \in A(\Omega)(\tilde{A}(\Omega))$ and every infinitesimal sequence of positive numbers $\{d_n\}$, there exists a sequence $\{m^n\}$ with entries in $A(\Omega_n)(\tilde{A}(\Omega_n))$ such that*

$$\acute{m}^n \rightarrow m^0,$$

$$E_0(m^0) = \lim_{n \rightarrow \infty} \acute{E}_n(\acute{m}^n)$$

- (iii) (**Compactness**) *Let $(d_n)_{n \in \mathbb{N}}$ be an infinitesimal sequence of positive numbers. Let $m^n \in A(\Omega_n)$ and let $\{\acute{E}_n(\acute{m}^n)\}$ be bounded. Then there exists a subsequence of $\{m^n\}$ (not relabeled) such that $(\acute{m}^n)_{n \in \mathbb{N}}$ converges to some $m^0 \in A_x(\Omega)$ in the sense of Definition 9.1.*

Proof. Lower semicontinuouty. We need to first prove some inequalities on A_{ω_n} , B_{ω_n} , and C_{ω_n} . Let us first prove that the numbers A and B are strictly positive (recall that $A_{\omega_n} = d_n^2 A_\omega$ and $B_{\omega_n} = d_n^2 B_\omega$.) Indeed, suppose for instance that $A = 0$ for some ω . Obviously the set $A_x(\Omega)$ is not empty. We fix a magnetization $m \in \tilde{A}_x(\Omega)$. We have

$$A_\omega = \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{k_2^2 + k_3^2} dk_2 dk_3 = 0$$

thus $a(k_2, k_3, \omega) = 0$ a.e. in \mathbb{R}^2 , $b(k_2, k_3, \omega) = \frac{k_2}{k_3} a(k_2, k_3, \omega) = 0$ a.e. in \mathbb{R}^2 . This means that

$$E_s(m) = 0 = \int_{\mathbb{R}^3} |\nabla u_s|^2 d\xi,$$

thus

$$\nabla u_s = 0 \quad \text{a.e. in} \quad \mathbb{R}^3.$$

According to (4) we have

$$\int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi d\xi = \int_{\partial\Omega_0} s \varphi d\xi \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^3)$$

thus $s = 0$ a.e. on $\partial\Omega$ which means $m_y = 0$ and $m_z = 0$ a.e. in Ω . Taking into account that m_x is a weakly differentiable function of one variable, m_x must be continuous in \mathbb{R} , therefore it must be identically 1 or -1 , which contradicts the fact $m \in \tilde{A}_x(\Omega)$. Let us consider three cases.

Case $C_\omega = 0$.

We can without loss of generality assume that the limit $\lim_{n \rightarrow \infty} \dot{E}(\dot{m}^n)$ exists and is finite. Thus $E(m^n) \leq M d_n^2$ for some constant M . According to Corollary (4.6) we have

$$E_{mag}(m) - E_{mag}(\bar{m}) = \delta_n \cdot d_n^2, \quad \text{where} \quad \lim_{n \rightarrow \infty} \delta_n = 0.$$

By virtue of Theorem 7.4 we have

$$\begin{aligned} M &\geq \frac{E(m^n)}{d_n^2} \geq \frac{E_{mag}(m^n)}{d_n^2} \\ &= \frac{E_{mag}(\bar{m}^n)}{d_n^2} + \delta_n \geq \frac{E_s(\bar{m}^n)}{d_n^2} + \delta_n \\ &\geq \frac{A_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_y^n(x)|^2 dx + \frac{B_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_z^n(x)|^2 dx \\ &\quad - 23u \int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) dx - \frac{M(A_\omega + B_\omega)}{\pi^2 |\omega|} d_n - \delta_n \\ &\geq \left(\frac{1}{2\pi^2} \min(A_\omega, B_\omega) - 23u \int_{\mathbb{R}} \right) (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) dx - \frac{M(A_\omega + B_\omega)}{\pi^2 |\omega|} d_n - \delta_n, \end{aligned}$$

thus

$$\int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) dx \leq \frac{4\pi^2 M}{\min(A_0, B_0)}$$

Therefore we obtain

$$\liminf_{n \rightarrow \infty} \frac{E_{mag}(m^n)}{d_n^2} \geq \liminf_{n \rightarrow \infty} \left(\frac{A_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_y^n(x)|^2 dx + \frac{B_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_z^n(x)|^2 dx \right) \quad (16)$$

It is standard that if $f^n \rightarrow f$ is the sense of Definition 9.1 then

$$\liminf_{n \rightarrow \infty} \|f_y^n\|_{L^2(\Omega)} \geq \|f_y\|_{L^2(\Omega)}, \quad \liminf_{n \rightarrow \infty} \|f_z^n\|_{L^2(\Omega)} \geq \|f_z\|_{L^2(\Omega)}.$$

Therefore utilizing Lemma 4.4 and (16) we establish

$$\liminf_{n \rightarrow \infty} \frac{E_{mag}(m^n)}{d_n^2} \geq \frac{A_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_y^0(x)|^2 dx + \frac{B_\omega}{2\pi^2} \int_{\mathbb{R}} |\bar{m}_z^0(x)|^2 dx. \quad (17)$$

From the weak convergence $\partial_x \dot{m}^n \rightharpoonup \partial_x m^0$ in $L^2(\Omega)$ we have

$$\liminf_{n \rightarrow \infty} \|\nabla \dot{m}^n\|_{L^2(\Omega)}^2 \geq \|\partial_x \dot{m}^0\|_{L^2(\Omega)}^2,$$

which gives together with (17),

$$\liminf_{n \rightarrow \infty} \dot{E}(\dot{m}^n) \geq E_0(m^0).$$

Case $C_\omega > 0$.

Let us first we prove that $C_\omega^2 < A_\omega B_\omega$. Denote

$$C_\omega^- = \int_0^{+\infty} \int_{-\infty}^0 \frac{k_3 |a|^2}{k_2(k_2^2 + k_3^2)} dk_2 dk_3, \quad C_\omega^+ = \int_0^{+\infty} \int_0^{+\infty} \frac{k_3 |a|^2}{k_2(k_2^2 + k_3^2)} dk_2 dk_3,$$

so that $C_\omega^- \leq 0$, $C_\omega^+ \geq 0$ and $C_\omega = C_\omega^- + C_\omega^+$. We have by the Schwartz inequality $|C_\omega^-|^2 \leq A_\omega B_\omega$, $|C_\omega^+|^2 \leq A_\omega B_\omega$, moreover in both cases the equality is not possible because as we saw before neither the ratio $\frac{a_\omega}{b_\omega}$ is constant, nor any of a_ω and b_ω is 0 a.e in the integration regions. Taking into account that $|C_\omega| \leq \max(|C_\omega^-|, |C_\omega^+|)$ we get $C_\omega^2 < A_\omega B_\omega$. We have furthermore for any t_n ,

$$\hat{m}_y^n \cdot \overline{\hat{m}_z^n} + \hat{m}_z^n \cdot \overline{\hat{m}_y^n} = \frac{1}{t_n} (|t_n \hat{m}_y^n + \hat{m}_z^n|^2 - t_n^2 |\hat{m}_y^n|^2 - |\hat{m}_z^n|^2),$$

thus

$$\begin{aligned} E_s^*(m^n) &= \frac{A_{\omega_n} - t_n C_{\omega_n}}{2\pi^2} \int_{\mathbb{R}} |m_y^n(x)|^2 dx + \frac{B_{\omega_n} - \frac{C_{\omega_n}}{t_n}}{2\pi^2} \int_{\mathbb{R}} |m_z^n(x)|^2 dx \\ &\quad + \frac{C_{\omega_n}}{2\pi^2 t_n} \int_{\mathbb{R}} |t_n m_y^n(x) + m_z^n(x)|^2 dx. \end{aligned}$$

Choose t_n such that

$$A_{\omega_n} - t_n C_{\omega_n} = B_{\omega_n} - \frac{C_{\omega_n}}{t_n} > 0$$

which is

$$t_n = \frac{A_{\omega_n} - B_{\omega_n} + \sqrt{(A_{\omega_n} - B_{\omega_n})^2 + 4C_{\omega_n}^2}}{2C_{\omega_n}}. \quad (18)$$

which is possible because $C_{\omega_n}^2 < A_{\omega_n} B_{\omega_n}$. Note that t_n does not depend on n , namely

$$t_n = \frac{A_\omega - B_\omega + \sqrt{(A_\omega - B_\omega)^2 + 4C_\omega^2}}{2C_\omega} := t_\omega.$$

Set $a_{\omega_n} = A_{\omega_n} - t_{\omega} C_{\omega_n} = d_n^2 (A_{\omega} - t_{\omega} C_{\omega})$. With this notation we have

$$E_s^*(m^n) = \frac{a_{\omega_n}}{2\pi^2} \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) dx + \frac{C_{\omega_n}}{2\pi^2 t_{\omega}} \int_{\mathbb{R}} |t_{\omega} m_y(x) + m_z(x)|^2 dx.$$

Like in the case $C_{\omega} = 0$ we can prove that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{E(m^n)}{d_n^2} &\geq \liminf_{n \rightarrow \infty} \frac{E_{ex}(m^n)}{d_n^2} + \liminf_{n \rightarrow \infty} \frac{E_{mag}(m^n)}{d_n^2} \\ &\geq |\omega| \int_{\mathbb{R}} |\partial_x m^0|^2 dx + \liminf_{n \rightarrow \infty} \frac{a_{\omega}}{2\pi^2} \int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) dx \\ &\quad + \liminf_{n \rightarrow \infty} \frac{C_{\omega}}{2\pi^2 t_{\omega}} \int_{\mathbb{R}} |t_{\omega} \bar{m}_y^n(x) + \bar{m}_z^n(x)|^2 dx \\ &\geq E_0(m^0) \end{aligned}$$

Case $C_{\omega} < 0$.

Note that formula (18) defines a negative t_{ω} , thus $\frac{C_{\omega_n}}{t_{\omega_n}} > 0$. Note furthermore that $a_{\omega} > 0$. The rest is analogous to the case $C_{\omega} > 0$.

Construction. As a candidate for m^n we take the constant sequence $m^n \equiv m^0$ in Ω_n . According to Theorem 7.4 we have for sufficiently big n

$$E_s(m^n) \leq E_s^*(m^n) + C(d_n^{\frac{13}{6}} + d_n E_{ex}(m^n)) \leq C d_n^2$$

for some constant C . Coupling this inequality with Lemma 7.6 we discover for big n

$$E(m^n) \leq C d_n^2. \quad (19)$$

In the lower-semi-continuity part we showed that if (19) is satisfied then for big n we have

$$\int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) dx \leq \frac{4C\pi^2}{a_{\omega}}.$$

thus utilizing again Theorem 7.4 we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{E_s(m^n)}{d_n^2} &\leq \limsup_{n \rightarrow \infty} \frac{E_s^*(m^n)}{d_n^2} \\ &= E_0(m^0) - |\omega| \int_{\mathbb{R}} |\partial_x m^0(x)|^2 dx. \end{aligned}$$

We have as well according to Lemma 7.6

$$0 \leq \lim_{n \rightarrow \infty} \frac{E_v(m^n)}{d_n^2} \leq \lim_{n \rightarrow \infty} M_{m^0} \cdot d_n(1 + d_n) = 0.$$

The last two inequalities complete the proof.

Compactness. We have that $E(m^n) \leq M d_n^2$, thus for big n we have

$$\|\partial_x \dot{m}^n\| \leq C, \quad \|\nabla_{yz} \dot{m}^n\| \leq C d_n,$$

for some constant C . Like in Lemma 8.1 we prove the relatively compactness of the sequence $\{\dot{m}^n\}$ with respect to the convergence defined in Definition

9.1. Assume a subsequence $\{m^n\}$ (not relabeled) converges to some m^0 . By $\|\nabla_{yz}\dot{m}^n\| \leq Cd_n$ we get that m^0 depends only on x , and by the lower semi-continuity part we have that

$$E_0(m^0) \leq \liminf_{n \rightarrow \infty} \dot{E}(\dot{m}^n) \leq M < \infty,$$

thus $m^0 \in A_x(\Omega)$. □

Corollary 9.3. *If a sequence of magnetizations $\{m^n\}$ satisfies $E(\dot{m}^n) \leq C$ for some constant C then*

$$E(m^n) \geq \int_{\Omega_n} |\nabla m^n|^2 + \frac{a_\omega}{2\pi^2|\omega|} \int_{\Omega_n} (|m_y^n|^2 + |m_z^n|^2) + o(d_n^2).$$

10 The minimal energy scaling and the rate of convergence

Next we find the minimal energy scaling. Recall that one can determine the minima of the energy functional

$$E_\alpha(m) = \int_{\mathbb{R}} |\partial_x m(x)|^2 dx + \alpha \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) dx$$

where $\alpha > 0$ and the admissible set is

$$\tilde{A}_0 = \{m: \mathbb{R} \rightarrow \mathbb{R}^3 : |m| = 1, m - \bar{e} \in H^1(\mathbb{R})\}.$$

By virtue of Lemma 4.7 and Remark 4.8 we have that $\lim_{n \rightarrow \pm\infty} m_x = \pm 1$, thus we can parameterize m as follows:

$$\begin{cases} m_x(x) = \sin \varphi(x) \\ m_y(x) = \cos \varphi(x) \cos \theta(x) \\ m_z(x) = \cos \varphi(x) \sin \theta(x) \end{cases}$$

where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in [0, 2\pi)$ and $\varphi(x) \rightarrow \pm \frac{\pi}{2}$ as $x \rightarrow \pm\infty$. It is clear that

$$\begin{aligned} E_\alpha(m) &= \int_{\mathbb{R}} \varphi'^2(x) + \theta'^2(x) \cos^2 \varphi(x) dx + \alpha \int_{\mathbb{R}} \cos^2 \varphi(x) dx \\ &\geq \int_{\mathbb{R}} \varphi'^2(x) dx + \alpha \int_{\mathbb{R}} \cos^2 \varphi(x) dx \\ &\geq 2\sqrt{\alpha} \int_{\mathbb{R}} |\varphi'(x)| |\cos \varphi(x)| dx \\ &\geq 2\sqrt{\alpha} \int_{\mathbb{R}} \varphi'(x) \cos \varphi(x) dx \\ &= 2\sqrt{\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t dt \\ &= 4\sqrt{\alpha} \end{aligned}$$

and the equality holds if and only if the following conditions hold:

$$\varphi'^2(x) = \alpha \cos^2 \varphi(x), \quad \varphi'(x) \cos \varphi(x) \geq 0,$$

$$\theta'(x) \cos^2 \varphi(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

The solution is then

$$\theta = \text{const}, \quad \varphi_{\alpha, \beta} = \arcsin \frac{e^{2\sqrt{\alpha}x} \cdot \beta - 1}{e^{2\sqrt{\alpha}x} \cdot \beta + 1}, \quad \text{where } \beta > 0.$$

For m we get

$$m^{\alpha, \beta} = \left(\frac{e^{2\sqrt{\alpha}x} \cdot \beta - 1}{e^{2\sqrt{\alpha}x} \cdot \beta + 1}, \frac{2\sqrt{\beta}e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta + 1} \cos \theta, \frac{2\sqrt{\beta}e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta + 1} \sin \theta \right). \quad (20)$$

Note that the minimal value of the energy does not depend on θ and for a fixed θ any minimizer can be obtained from $m^\alpha := m^{\alpha, 1}$ by translation in the x direction. The minimizer m^α satisfies $m_x^\alpha(0) = 0$. The minimal value of E_α in \tilde{A}_0 will be $4\sqrt{\alpha}$. Let us now find the minimal value of the reduced problem for any ω . Observe that the limit energy

$$E_0(m) \geq |\omega| \int_{\mathbb{R}} |\partial_x m(x)|^2 dx + \frac{a_\omega}{2\pi^2} \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) dx \geq \frac{2\sqrt{2|\omega|a_\omega}}{\pi},$$

and the minimum is realized by an m^{α_ω} satisfying

$$\alpha_\omega = \frac{a_\omega}{2\pi^2|\omega|} \quad \text{and} \quad t_\omega m_y^{\alpha_\omega} + m_z^{\alpha_\omega} = 0, \quad x \in \mathbb{R},$$

thus $\theta_\omega = -\arctan t_\omega$. All other minimizers of the limit energy $E_0(m)$ are obtained by translations of m^{α_ω} .

Denote

$$m^\omega = \left(\frac{e^{2\sqrt{\alpha_\omega}x} - 1}{e^{2\sqrt{\alpha_\omega}x} + 1}, \frac{2e^{\sqrt{\alpha_\omega}x}}{e^{2\sqrt{\alpha_\omega}x} + 1} \cos \theta_\omega, \frac{2e^{\sqrt{\alpha_\omega}x}}{e^{2\sqrt{\alpha_\omega}x} + 1} \sin \theta_\omega \right),$$

$$\varphi_\omega(x) = \arcsin \frac{e^{2\sqrt{\alpha_\omega}x} - 1}{e^{2\sqrt{\alpha_\omega}x} + 1}, \quad \text{and} \quad \theta_\omega = \arctan t_\omega. \quad (21)$$

The minimum of the limit energy is then

$$E_0^{min} = \frac{2\sqrt{2|\omega|a_\omega}}{\pi}.$$

Next we denote

$$E_n^{min} = \min_{m \in \tilde{A}(\Omega_n)} E(m),$$

and show that

$$\lim_{n \rightarrow \infty} \frac{E_n^{min}}{d_n^2} = E_0^{min}.$$

We will actually prove a rate of convergence.

Theorem 10.1. *For sufficiently big n the following bound holds:*

$$\left| \frac{E_n^{min}}{d_n^2} - E_0^{min} \right| \leq 220\pi^2 \sqrt{\frac{2|\omega|}{a_\omega}} (\text{per}(\omega))^2 d_n^{\frac{1}{6}}.$$

Proof. We have that $\Omega_n = \mathbb{R} \times (d_n \cdot \omega)$. Consider as usual the constant sequence of magnetizations $m^n \equiv m^\omega$ regarding m^n as a magnetization defined in Ω_n , where m^ω is the minimizer in (21). It is clear that

$$m^n \in \tilde{A}_n \quad \text{and} \quad E_n^{min} \leq E(m^n).$$

We have that

$$E_{ex}(m^n) = c_\omega d_n^2 \int_{\mathbb{R}} |\partial_x m^\omega|^2 dx.$$

and according to Lemma 7.6

$$E_v(m^n) \leq M_{m^\omega} d_n^3 (1 + d_n). \quad (22)$$

m^ω is the minimizer of the limit energy E_0 , thus

$$|\omega| \int_{\mathbb{R}} |\partial_x m^\omega|^2 dx = \frac{\sqrt{2|\omega|a_\omega}}{\pi}, \quad \int_{\mathbb{R}} (|m_y^\omega|^2 + |m_z^\omega|^2) dx = 2\pi \sqrt{\frac{2|\omega|}{a_\omega}}.$$

We have that

$$E_s^*(m^n) = \frac{d_n^2 a_\omega}{2\pi^2} \int_{\mathbb{R}} (|m_y^\omega|^2 + |m_z^\omega|^2) dx = \frac{d_n^2 \sqrt{2|\omega|a_\omega}}{\pi},$$

hence by virtue of Theorem 7.4 we get

$$E_s(m^n) \leq E_s^*(m^n) + 46\pi \sqrt{\frac{2|\omega|}{a_\omega}} \cdot u_n + \frac{d_n^3 (A_\omega + B_\omega) \sqrt{2a_\omega}}{\pi^3 \sqrt{|\omega|}}.$$

Recall that $u_n = d_n^{\frac{1}{6}} (\text{per}(\omega_n))^2 = d_n^{\frac{13}{6}} (\text{per}(\omega))^2$, therefore from last inequality and (22) we obtain for big n ,

$$\frac{E(m^n)}{d_n^2} - \frac{2\sqrt{2|\omega|a_\omega}}{\pi} \leq 50\pi \sqrt{\frac{2c_\omega}{a_\omega}} (\text{per}(\omega))^2 d_n^{\frac{1}{6}}, \quad (23)$$

Assume now $m^n \in \tilde{A}(\Omega_n)$ is an energy minimizer. Then by (23) we have for bin n that, $E(m^n) \leq M d_n^2$, with $M = \frac{3\sqrt{2|\omega|a_\omega}}{\pi}$. By virtue of Theorem 7.4,

$$M d_n^2 \geq \left(\frac{a_{\omega_n}}{2\pi^2} - 23u_n \right) \int_{\mathbb{R}} (|\bar{m}_y^n|^2 + |\bar{m}_z^n|^2) dx - \frac{M(A_{\omega_n} + B_{\omega_n})d_n}{\pi^2 |\omega|},$$

thus we get for big n

$$\int_{\mathbb{R}} (|\bar{m}_y^n|^2 + |\bar{m}_z^n|^2) dx \leq \frac{3\pi^2 M}{a_\omega}.$$

Applying now Theorem 7.4 again and taking into account the last inequality we establish for big n

$$E_s(\bar{m}^n) \geq E_s^*(\bar{m}^n) - \frac{70\pi^2 M}{a_\omega} \cdot u_n.$$

For the energy functional of \bar{m}^n we obtain

$$E(\bar{m}^n) \geq E_{ex}(\bar{m}^n) + E_s^*(\bar{m}^n) - 210\pi^2 \sqrt{\frac{2|\omega|}{a_\omega}} \cdot u_n$$

Recall that the vector $\lim_{x \rightarrow \pm\infty} \bar{m}^n(x) = \pm 1$, hence by the argument done for the functional $E_\alpha(m)$ we establish

$$\frac{E(\bar{m}^n)}{d_n^2} - \frac{2\sqrt{2|\omega|a_\omega}}{\pi} \geq -210\pi^2 \sqrt{\frac{2|\omega|}{a_\omega}} \cdot (\text{per}(\omega))^2 d_n^{\frac{1}{6}}. \quad (24)$$

By Lemmas 4.3 and 4.5 we have that if $E(m^n) \leq Md_n^2$ then

$$\frac{E_{mag}(m^n)}{d_n^2} - \frac{E_{mag}(\bar{m}^n)}{d_n^2} = O(d_n)$$

as n goes to ∞ , thus

$$E(m^n) \geq E_{ex}(\bar{m}^n) + E_{mag}(\bar{m}^n) + O(d_n^3) = E(\bar{m}^n) - O(d_n^3).$$

Finally taking into account (24) and (23) we discover for big n ,

$$\left| \frac{E(m^n)}{d_n^2} - \frac{2\sqrt{2|\omega|a_\omega}}{\pi} \right| \leq 220\pi^2 \sqrt{\frac{2|\omega|}{a_\omega}} \cdot (\text{per}(\omega))^2 d_n^{\frac{1}{6}}$$

□

11 Convergence of almost minimizers

Throughout this section we will consider a sequence of domain-magnetization-energy triples $(\Omega_n, m^n, E(m^n))_{n \in \mathbb{N}}$ such that $\Omega_n = \mathbb{R} \times (d_n \cdot \omega)$, $m^n \in \tilde{A}(\Omega_n)$, $d_n \rightarrow 0$ and

$$\lim_{n \rightarrow \infty} \frac{E(m^n)}{d_n^2} = E_0^{min}. \quad (25)$$

We will call such a sequence almost minimizing.

Lemma 11.1. *If $\{\dot{m}^n\}$ converges to some $m^0 \in \tilde{A}_x(\Omega)$ in the sense of Definition 9.1 then*

- (i) $\lim_{n \rightarrow \infty} \|\nabla \dot{m}^n\|_{L^2(\Omega)} = \|\nabla m^0\|_{L^2(\Omega)},$
- (ii) $\lim_{n \rightarrow \infty} \|\dot{m}_y^n\|_{L^2(\Omega)} = \|m_y^0\|_{L^2(\Omega)},$
- (iii) $\lim_{n \rightarrow \infty} \|\dot{m}_z^n\|_{L^2(\Omega)} = \|m_z^0\|_{L^2(\Omega)}.$

Proof. We have already shown that the above limits with \liminf are big or equal than the corresponding expected limits, thus it remains to only show the opposite inequalities with \limsup . Since $E(m^n) \leq M d_n^2$, then we have by Lemma 4.6,

$$\lim_{n \rightarrow \infty} \frac{E_{mag}(m^n)}{d_n^2} = \lim_{n \rightarrow \infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2}.$$

Assume now in contradiction that one of the three inequalities fails. Therefore we have for some $\delta > 0$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{E(m^n)}{d_n^2} &\geq \max \left(\limsup_{n \rightarrow \infty} \|\partial_x \dot{m}^n\|_{L^2(\Omega)}^2 + \liminf_{n \rightarrow \infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2}, \right. \\ &\quad \left. \liminf_{n \rightarrow \infty} \|\partial_x \dot{m}^n\|_{L^2(\Omega)}^2 + \limsup_{n \rightarrow \infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2} \right) \\ &\geq \int_{\Omega_0} |\partial_x m^0(\xi)|^2 d\xi + \frac{a_{\omega_0}}{2\pi^2} \int_{\mathbb{R}} (|m_y^0(x)|^2 + |m_z^0(x)|^2) dx + \delta \\ &\geq \frac{2\sqrt{2}|\omega|a_\omega}{\pi} + \delta, \end{aligned}$$

which contradicts (25). □

Corollary 11.2. *Let $\{m^n\}$ and m^0 be as in Lemma 11.1. Then*

- (i) $\lim_{n \rightarrow \infty} \|\dot{m}_y^n\|_{L^2(\Omega)} = \|m_y^0\|_{L^2(\Omega)},$
- (ii) $\lim_{n \rightarrow \infty} \|\dot{m}_z^n\|_{L^2(\Omega)} = \|m_z^0\|_{L^2(\Omega)}.$

Proof. It follows from Lemmas 11.1 and 4.4. □

Lemma 11.3. *Let $\{m^n\}$ and m^0 be as in Lemma 11.1. Then*

- (i) $\lim_{n \rightarrow \infty} \|\nabla \dot{m}^n - \nabla m^0\|_{L^2(\Omega)} = 0$
- (ii) $\lim_{n \rightarrow \infty} \|\dot{m}_y^n - m_y^0\|_{L^2(\Omega)} = 0, \quad \lim_{n \rightarrow \infty} \|\dot{m}_z^n - m_z^0\|_{L^2(\Omega)} = 0.$

Proof. It is clear that $\|\nabla \dot{m}^n\|_{L^2(\Omega)} \geq \|\nabla \dot{\bar{m}}^n\|_{L^2(\Omega)}$ and note that if we had a strict inequality $\limsup_{n \rightarrow \infty} \|\nabla \dot{m}^n\|_{L^2(\Omega)} > \limsup_{n \rightarrow \infty} \|\nabla \dot{\bar{m}}^n\|_{L^2(\Omega)}$ then it would contradict the first statement of Lemma 11.1. Therefore $\|\nabla \dot{m}^n\|_{L^2(\Omega)} = \|\nabla \dot{\bar{m}}^n\|_{L^2(\Omega)}$ and from the weak convergence $\dot{m}^n \rightharpoonup m^0$ we get (i). Fix now $l > 0$. We have by virtue of Corollary 11.2,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \int_{\Omega} |\dot{m}_y^n - m_y^0|^2 &\leq \limsup_{n \rightarrow \infty} \int_{[-l, l] \times \omega} |\dot{m}_y^n - m_y^0|^2 \\
&\quad + \limsup_{n \rightarrow \infty} \int_{\Omega \setminus ([-l, l] \times \omega)} |\dot{m}_y^n - m_y^0|^2 \\
&\leq 2 \limsup_{n \rightarrow \infty} \int_{\Omega \setminus ([-l, l] \times \omega)} (|\dot{m}_y^n|^2 + |m_y^0|^2) \\
&\leq 2 \limsup_{n \rightarrow \infty} \int_{\Omega} (|\dot{m}_y^n|^2 + |m_y^0|^2) \\
&\quad - 2 \liminf_{n \rightarrow \infty} \int_{[-l, l] \times \omega} (|\dot{m}_y^n|^2 + |m_y^0|^2) \\
&= 4|\omega| \int_{\mathbb{R} \setminus [-l, l]} |m_y^0(x)|^2 dx.
\end{aligned}$$

From the arbitrariness of l we get the validity of (ii). The proof of (iii) is straightforward. \square

Lemma 11.4. *Let $\{m^n\}$ and m^0 be as in Lemma 11.1. Assume in addition that for some $N \in \mathbb{N}$ and $l > 0$ we have for all $n \geq N$*

$$\bar{m}^n(x) \leq 0, \quad x \in (-\infty, -l] \quad \text{and} \quad \bar{m}^n(x) \geq 0, \quad x \in [l, +\infty).$$

Then

$$\lim_{n \rightarrow \infty} \|\dot{m}^n - m^0\|_{H^1(\Omega)} = 0.$$

Proof. By Lemma 11.3 it suffice to show that $\lim_{n \rightarrow \infty} \|\dot{m}_x^n - m_x^0\|_{L^2(\Omega)} = 0$. Since $m^0 \in \tilde{A}_x(\Omega)$ there exists $l_1 > 0$ such that

$$m_x^0(x) \leq -\frac{1}{2}, \quad x \in (-\infty, l_1] \quad \text{and} \quad m_x^0(x) \geq \frac{1}{2}, \quad x \in [l_1, +\infty).$$

For any fixed $l_2 > \max(l, l_1)$ we have that

$$\int_{\Omega} |\dot{m}_x^n - m_x^0|^2 = \int_{[-l_2, l_2] \times \omega} |\dot{m}_x^n - m_x^0|^2 + \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}_x^n - m_x^0|^2.$$

The first summand converges to zero and we have furthermore that $\|\dot{m}_x^n - \dot{m}_x^n\|_{L^2(\Omega)} \rightarrow 0$, thus it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}_x^n - m_x^0|^2 = 0.$$

For $n \geq N$ we have

$$\begin{aligned}
&\int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}_x^n - m_x^0|^2 \leq \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} \left| |\dot{m}_x^n|^2 - |m_x^0|^2 \right| \\
&\leq \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} \left| |\dot{m}_x^n|^2 - |m_x^n|^2 \right| + \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} \left| |\dot{m}_x^n|^2 - |m_x^0|^2 \right|.
\end{aligned}$$

The first summand converges to zero, for the second summand we have by Lemma 11.2

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} & \left| |\dot{m}_x^n|^2 - |m_x^0|^2 \right| \\
& \leq \limsup_{n \rightarrow \infty} \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} (|\dot{m}_y^n|^2 + |\dot{m}_z^n|^2 + |m_y^0|^2 + |m_z^0|^2) \\
& \leq 2 \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} (|m_y^0|^2 + |m_z^0|^2),
\end{aligned}$$

which converges to zero as l_2 goes to infinity. \square

Lemma 11.5. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Then for any interval $(a, b) \subset \mathbb{R}$, positive α and a unit vector field $f \in H^1((a, b) \times \omega, \mathbb{R}^3)$ the following inequality holds:*

$$\int_{(a, b) \times \omega} |\partial_x f|^2 + \alpha^2 \int_{(a, b) \times \omega} (|f_y|^2 + |f_z|^2) \geq 2\alpha |\omega| |\bar{f}_x(a) - \bar{f}_x(b)|.$$

(The endpoints a and b can take values $-\infty$ and $+\infty$ respectively).

Proof. Fix a point $(y, z) \in \omega$ and consider the vector field f on the segment with endpoints (a, y, z) and (b, y, z) . Being an H^1 vector field, it must be absolutely continuous on that segment as a function of one variable, thus denoting

$$f_x(x, y, z) = \sin \varphi(x), f_y(x, y, z) = \cos \varphi(x) \cos \theta(x), f_z(x, y, z) = \cos \varphi(x) \sin \theta(x)$$

we obtain that φ and θ are differentiable in $[a, b]$ a.e.. Therefore s we can calculate

$$\begin{aligned}
& \int_{(a, b) \times (y, z)} |\partial_x f(\xi)|^2 dx + \alpha^2 \int_{(a, b) \times (y, z)} (|f_y(\xi)|^2 + |f_z(\xi)|^2) dx \\
& = \int_a^b (\varphi'^2(x) + \theta'^2(x) \cos^2 \varphi(x)) dx + \alpha^2 \int_a^b \cos^2 \varphi(x) dx \\
& \geq \int_a^b (\varphi'^2(x) dx + \alpha^2 \int_a^b \cos^2 \varphi(x) dx \\
& \geq 2\alpha \left| \int_a^b \varphi'(x) \cos \varphi(x) dx \right| \\
& = 2\alpha |f_x(a, y, z) - f_x(b, y, z)|.
\end{aligned}$$

An integration of the obtained inequality over ω completes the proof. \square

Lemma 11.6. *Let the sequence of intervals $([b_n^1, b_n^2])_{n \in \mathbb{N}}$ be such that*

$$\bar{m}_x^n(b_n^1) = -\frac{1}{2}, \quad \bar{m}_x^n(b_n^2) = \frac{1}{2}, \quad |\bar{m}_x^n(x)| \leq \frac{1}{2}, \quad x \in [b_n^1, b_n^2].$$

Then for sufficiently big n we have

$$\bar{m}_x^n(x) < -\frac{1}{3}, \quad x \in (-\infty, b_n^1] \quad \text{and} \quad \bar{m}_x^n(x) > \frac{1}{3}, \quad x \in [b_n^2, +\infty).$$

Proof. Assume in contradiction that for a subsequence (not relabeled) there is a point $b_n^3 \in (-\infty, b_n^1)$ such that $\bar{m}_x^n(b_n^3) \geq -\frac{1}{3}$. Since $\bar{m}_x^n(-\infty) = -1$ and \bar{m}_x^n is continuous we can without loss of generality assume that $\bar{m}_x^n(b_n^3) = -\frac{1}{3}$. Utilizing Lemma 11.5 for the intervals $(-\infty, b_n^3]$, $[b_n^3, b_n^1]$, $[b_n^1, +\infty)$ and Corollary 9.3 we get

$$\begin{aligned} E(m^n) &\geq \int_{\Omega_n} |\nabla m^n|^2 + \frac{a_\omega}{2\pi^2|\omega|} \int_{\Omega_n} (|m_y^n|^2 + |m_z^n|^2) + o(d_n^2) \\ &\geq \frac{2}{\pi} \sqrt{\frac{a_\omega}{2|\omega|}} |\omega_n| \left(\left| -1 + \frac{1}{3} \right| + \left| -\frac{1}{3} + \frac{1}{2} \right| + \left| -\frac{1}{2} - 1 \right| \right) + o(d_n^2) \\ &= \frac{7\sqrt{2a_\omega|\omega|}}{3\pi} d_n^2 + o(d_n^2), \end{aligned}$$

thus

$$\liminf_{n \rightarrow \infty} \frac{E(m^n)}{d_n^2} \geq \frac{7}{6} E_{min}^0$$

which is a contradiction. \square

Theorem 11.7. *Assume that the domain ω is so that $C_\omega^2 + (A_\omega - B_\omega)^2 > 0$. Then for any sequence of magnetizations $\{m^n\} \subset \tilde{A}(\Omega_n)$, satisfying (25) there exist a sequence $\{T_n\}$ of translations in the x direction and a sequence $\{R_n\}$ of rotations in the OYZ plane, each of which is either the identity or the rotation by 180 degree such that the sequence with the terms $\tilde{m}^n(x, y, z) = m^n(T_n(R_n(x, y, z)))$ converges to m^ω in the sense of Definition 9.1.*

Proof. First of all note that the change of variables mentioned in the theorem translate the domain Ω to itself and preserve the energy, which means that the minimization problem (1) is invariant under that kind of transforms. Let the intervals $[b_n^1, b_n^2]$ be as in Lemma 11.6. We prove the theorem by constructing such sequences. In the first step we prove that if a sequence of magnetizations converges to some $m^0 \in \tilde{A}(\Omega)$ in the sense of Definition 9.1 and satisfies the conditions $E(m^n) \leq M d_n^2$ and $\bar{m}_y^n(x_0) \geq 0$ for some $x_0 \in \mathbb{R}$, $M > 0$ and for big n then $m_y^0(x_0) \geq 0$. Assume in contradiction that $m_y^0(x_0) = \delta < 0$.

$$\lim_{n \rightarrow \infty} \int_{x_0-1}^{x_0+1} |\bar{m}_y^n(x) - m_y^0(x)|^2 dx = 0. \quad (26)$$

On the other hand we have

$$\int_{\Omega_n} |\partial_x \bar{m}^n|^2 \leq \int_{\Omega_n} |\partial_x m^n|^2 \leq M d_n^2,$$

thus

$$\int_{\mathbb{R}} |\partial_x \bar{m}^n(x)|^2 dx \leq \frac{M}{|\omega|} = M_1,$$

which yields that the sequence $\{\bar{m}^n(x)\}$ is equicontinuous in \mathbb{R} , thus there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$,

$$\bar{m}_y^n(x) \geq \frac{\delta}{3}, \quad x \in [x_0 - \epsilon, x_0 + \epsilon].$$

By the continuity of m^0 we can assume that,

$$m_y^0(x) \leq \frac{2\delta}{3} \quad x \in [x_0 - \epsilon, x_0 + \epsilon].$$

Combining the last inequality with the inequality for \bar{m}_y^n we obtain

$$\int_{x_0-1}^{x_0+1} |\bar{m}_y^n(x) - m_y^0(x)|^2 dx \geq 2\frac{\delta^2}{9} \min(\epsilon, 1)$$

which contradicts (26). The same sing preserving property can be also proved for the first and the third components of \bar{m}^n and also for the opposite sign. This means in particular that if $\bar{m}_x^n(x_0) = 0$ for big n then $m_x^0(x_0) = 0$. In the second step we construct the sequences $\{T_n\}$ and $\{R_n\}$. Let the intervals $[b_n^1, b_n^2]$ be as in Lemma 11.6 and $x_n \in [b_n^1, b_n^2]$ be such that $\bar{m}_x^n(x_n) = 0$. By continuity such intervals and points exist for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ we choose T_n to be the translation by x_n and the rotation R_n to be the identity if $\bar{m}_y^n(x_n) \geq 0$ and the rotation by 180 degree otherwise. We show now that the sequence $(\tilde{m}^n)_{n \in \mathbb{N}}$ converges to some $m^0 \in \tilde{A}(\Omega)$ in the sense of Definition 9.1. Utilizing the Γ -convergence theorem we get that the sequence $\{\tilde{m}^n\}$ is relatively compact thus what we have to actually show now is that its every convergent subsequence has the same limit. Suppose $\{\tilde{m}^{n_k}\}$ converges to some $m^0 \in X(\Omega)$. By lower semi-continuity m^0 has finite reduced energy, thus $m^0 \in A(\Omega)$. We show that actually $m^0 \in \tilde{A}(\Omega)$. By virtue of Lemma 4.7 there exists $C > 0$ such that $b_{n_k}^2 - b_{n_k}^1 \leq C$ for any $k \in \mathbb{N}$, therefore by Lemma 11.6 we obtain that $\tilde{m}_x^{n_k}$ is negative in $(-\infty, -C]$ and is positive in $[C, +\infty)$ and hence using the fact that \tilde{m}^{n_k} converges to m_0 in $L_{loc}^2(\mathbb{R})$ we get that m_0 must be nonpositive in $(-\infty, -C]$ and be nonnegative in $[C, +\infty)$ and therefore $m^0 \in \tilde{A}(\Omega)$. Now the above proved fact states that $m_x^0(0) = 0$ and $m_y^0(0) \geq 0$. Furthermore by the lower semi-continuity we have that

$$E_0(m^0) \leq \liminf_{n \rightarrow \infty} \frac{E(\tilde{m}^{n_k})}{d_n^2} = \liminf_{n \rightarrow \infty} \frac{E(m^{n_k})}{d_n^2} = E_{min}^0$$

thus m^0 is a minimizer of E_0 . It is easy to see now that the properties $m_x(0) = 0$ and $m_y(0) \geq 0$ determine m^0 in the unique way, namely we get $m^0 = m^\omega$ as claimed. \square

Theorem 11.8. *Assume that the domain ω is so that $C_\omega^2 + (A_\omega - B_\omega)^2 > 0$. Then for any sequence of magnetizations $\{m^n\}$ satisfying (25) there exist a sequence $\{T_n\}$ of translations in the x direction and a sequence $\{R_n\}$ of rotations in the OYZ plane, each of which is either the identity or the rotation by 180 degrees such that for $\tilde{m}^n(x, y, z) = m^n(T_n(R_n(x, y, z)))$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \|\tilde{m}^n - m^\omega\|_{H^1(\Omega_n)} = 0.$$

Proof. It is a consequence of Lemmas 11.4, 11.6 and Theorem 11.7. \square

Corollary 11.9. *Theorem 11.8 holds for any sequence of minimizers $\{m^n\}$.*

We mention that it is easy to see that any rectangle that is not a square and any ellipse that is not a circle satisfies the condition

$$C_\omega^2 + (A_\omega - B_\omega)^2 > 0.$$

This condition shows that the cross section ω does not have many rotational symmetries in some sense. For instance, if ω has two perpendicular axis of symmetry, then one can show that $C_\omega = 0$. It is also worth mentioning that one can prove a modified version of Theorem 11.8 in the case when ω is a disc or a canonical polygon, namely due to the symmetry one can not state any of the rotations R_n is either the identity the rotation by 180 degree, but one can prove their existence. In conclusion we state that Theorem 11.8 shows that in thin wires energy minimizers with a 180 degree domain wall are transverse walls that have the shape of m^ω .

A Appendix

The following two identities are well known and can be found in most calculus and analysis books:

$$\int_0^{+\infty} \frac{\cos px}{x^2 + q^2} dx = \frac{\pi}{2q} e^{-pq}, \quad q > 0, p > 0 \quad (27)$$

$$\int_0^{+\infty} \frac{e^{-p_1 x} \cos q_1 x - e^{-p_2 x} \cos q_2 x}{x} dx = \frac{1}{2} \ln \frac{p_1^2 + q_1^2}{p_2^2 + q_2^2}, \quad p_1, p_2 > 0, q_1, q_2 \in \mathbb{R} \quad (28)$$

Lemma A.1. *For any $p, q, l > 0$ the following inequality holds:*

$$\left| \int_l^{+\infty} \frac{\sin qt}{t} e^{-pt} dt \right| \leq \pi.$$

Proof. Making $t = \frac{x}{q}$ change of variables and denoting $r = \frac{p}{q}$, $L = ql$ we get

$$\int_l^{+\infty} \frac{\sin qt}{t} e^{-pt} dt = \int_L^{+\infty} \frac{\sin x}{x} e^{-rx} dx$$

Assume $L \in [k\pi, (k+1)\pi]$ for some k . Evidently

$$\left| \int_L^{+\infty} \frac{\sin x}{x} e^{-rx} dx \right| \leq \max \left(\left| \int_{x_k}^{\infty} \frac{\sin x}{x} e^{-rx} dx \right|, \left| \int_{x_{k+1}}^{\infty} \frac{\sin x}{x} e^{-rx} dx \right| \right),$$

thus it suffice to prove the lemma for $L = k\pi$ for some k . We expand the integral in the following way:

$$\begin{aligned} \int_{k\pi}^{+\infty} \frac{\sin x}{x} e^{-rx} dx &= \sum_{i=k}^{\infty} \int_{i\pi}^{(i+1)\pi} \frac{\sin x}{x} e^{-rx} dx \\ &= \sum_{i=k}^{\infty} \int_0^{\pi} \frac{(-1)^i \sin t}{t + \pi i} e^{-r(t+\pi i)} dt \\ &= \int_0^{\pi} \sin t \sum_{i=k}^{\infty} \frac{(-1)^i}{t + \pi i} e^{-r(t+\pi i)} dt. \end{aligned}$$

For a fixed t we have an alternating series with decreasing terms with their absolute value, therefore

$$\left| \int_{k\pi}^{+\infty} \frac{\sin x}{x} e^{-rx} dx \right| \leq \int_0^\pi \frac{\sin t}{t + \pi k} e^{-r(t+\pi k)} dt \leq \int_0^\pi \frac{\sin t}{t} dt \leq \pi.$$

□

Lemma A.2. *For any $p \geq 0$ the function*

$$I_p(y) = \int_0^{+\infty} \frac{\sin pt}{t^2 + y^2} dt$$

is nonnegative and decreasing in y in $(0, +\infty)$ and

$$I_p(y) \leq \frac{7p^{\frac{1}{3}}}{y^{\frac{2}{3}}}.$$

Proof. The case $p = 0$ is evident. Suppose now $p > 0$. We make a change of variables $t = \frac{x}{p}$ to get

$$I(p, y) = p \int_0^{+\infty} \frac{\sin x dx}{x^2 + p^2 y^2} = p I_1(py).$$

We consider now $I_1(y)$ for $y > 0$. We have

$$\begin{aligned} I_1(y) &= \int_0^{+\infty} \frac{\sin t}{t^2 + y^2} dt = \sum_{n=0}^{\infty} \int_0^{2\pi} \frac{\sin t}{(t + 2\pi n)^2 + y^2} dt \\ &= \int_0^\pi \sin t \cdot \sum_{n=0}^{\infty} \frac{2\pi(t + 2\pi n) + \pi^2}{((t + 2\pi n)^2 + y^2)((t + \pi(2n + 1))^2 + y^2)} dt. \end{aligned}$$

It is now evident that $I_1(y)$ is nonnegative and decreasing in y in $(0, +\infty)$ and therefore the same does $I_p(y)$. Note that for any $n \geq 1$ and $t \in [0, \pi]$ one can easily prove using the Cauchy inequality, that

$$\frac{2\pi(t + 2\pi n) + \pi^2}{((t + 2\pi n)^2 + y^2)((t + \pi(2n + 1))^2 + y^2)} < \frac{\pi^2(4n + 3)}{4\pi^2 n^2(4\pi^2 n^2 + y^2)} < \frac{1}{9n^2 y^{\frac{2}{3}}},$$

hence

$$\begin{aligned} I(1, y) &< \int_0^\pi \frac{\sin t(\pi^2 + 2\pi t)}{(t^2 + y^2)((t + \pi)^2 + y^2)} dt + \sum_{n=1}^{\infty} \frac{1}{9n^2 y^{\frac{2}{3}}} \int_0^\pi \sin t dt \\ &< \int_0^\pi \frac{3\pi^2 t}{3\pi^2 \left(\frac{t^4 y^2}{4}\right)^{\frac{1}{3}}} dt + \frac{4}{9y^{\frac{2}{3}}} \\ &< \frac{7}{y^{\frac{2}{3}}}. \end{aligned}$$

Finally we have

$$I_p(y) = p I_1(py) < \frac{7p}{(py)^{\frac{2}{3}}} = \frac{7p^{\frac{1}{3}}}{y^{\frac{2}{3}}}$$

□

Lemma A.3. For any $p > 0, l \geq 0$ and a decreasing function $f \in C((0, +\infty), \mathbb{R}^+)$ such that $\lim_{x \rightarrow \infty} f(x) = 0$, the following bounds hold:

$$\left| \int_l^{+\infty} f(t) \cos pt \, dt \right| \leq \frac{4f(l)}{p}, \quad \left| \int_l^{+\infty} f(t) \sin pt \, dt \right| \leq \frac{4f(l)}{p}.$$

Proof. Denote $t_n = \frac{\pi n}{p}$, $n \in \mathbb{N}$ and assume $l \in [t_m, t_{m+1}]$. In any interval $[t_n, t_{n+1}]$ the function $\sin pt$ has a constant sign, therefore by the intermediate value theorem we have for some points $t'_n \in [t_n, t_{n+1}]$,

$$\begin{aligned} \left| \int_l^{+\infty} f(t) \sin pt \, dt \right| &\leq \left| \int_l^{t_{m+1}} f(t) \sin pt \, dt \right| + \left| \int_{t_{m+1}}^{+\infty} f(t) \sin pt \, dt \right| \\ &\leq f(l) \left| \int_{t_m}^{t_{m+1}} \sin pt \, dt \right| + \left| \frac{2}{p} \sum_{k=m+1}^{\infty} (-1)^k f(t'_k) \right| \\ &\leq \frac{2f(l)}{p} + \frac{2f(t_{m+1})}{p} \\ &\leq \frac{4f(l)}{p}. \end{aligned}$$

The first integral can be estimated in the same way. □

Lemma A.4. For any $p, q, l > 0$ the following bounds hold:

$$\begin{aligned} \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos px \cos qy}{x^2 + y^2} \, dx \, dy \right| &\leq \frac{2\pi}{ql}, \\ \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\sin px \sin qy}{x^2 + y^2} \, dx \, dy \right| &\leq \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}, \\ \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos px \sin qy}{x^2 + y^2} \, dx \, dy \right| &\leq \frac{\pi^2}{2}, \\ \left| \int_l^{+\infty} \int_0^{+\infty} \frac{\sin px \cos qy}{x^2 + y^2} \, dx \, dy \right| &\leq \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}. \end{aligned}$$

Proof. We apply (27) and Lemmas A.2, A.3 to get

$$\left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos px \cos qy}{x^2 + y^2} \, dx \, dy \right| = \frac{\pi}{2} \left| \int_l^{+\infty} \frac{e^{-py} \cos qy}{y} \, dy \right| \leq \frac{\pi}{2} \cdot \frac{4e^{-pl}}{ql} < \frac{2\pi}{ql}.$$

For the second and the forth integrals we again apply Lemmas A2 and A3,

$$\left| \int_l^{+\infty} \int_0^{+\infty} \frac{\sin px \sin qy}{x^2 + y^2} \, dx \, dy \right| = \left| \int_l^{+\infty} I_p(y) \sin qy \, dy \right| \leq \frac{4I_p(l)}{q} \leq \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}.$$

Similarly

$$\left| \int_l^{+\infty} \int_0^{+\infty} \frac{\sin px \cos qy}{x^2 + y^2} \, dx \, dy \right| \leq \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}.$$

For the third integral we utilize (27) and Lemma A.1,

$$\left| \int_l^{+\infty} \int_0^{+\infty} \frac{\cos px \sin qy}{x^2 + y^2} \, dx \, dy \right| \leq \frac{\pi}{2} \left| \int_l^{+\infty} \frac{e^{-py} \sin qy}{y} \, dy \right| \leq \frac{\pi^2}{2}.$$

□

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